Preliminary material (mathematical requirements)

Vector Analysis

Vector algebra:

Addition; Subtraction; Multiplication

Vector Calculus:

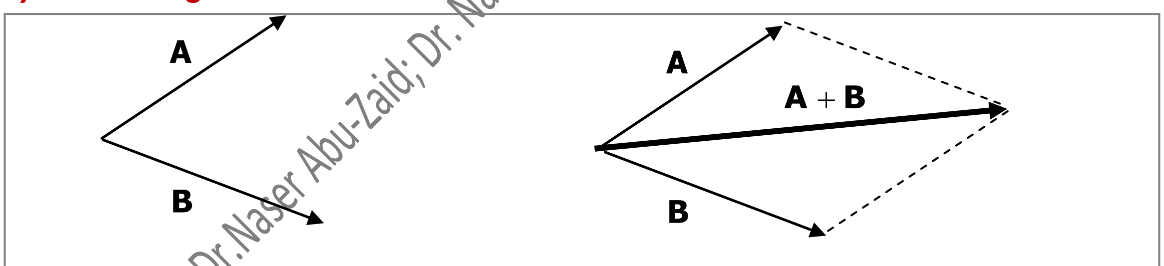
Differentiation; Integration

Vector: A quantity with both magnitude and direction. (Force $\mathbf{F} = 10N$ to the east).

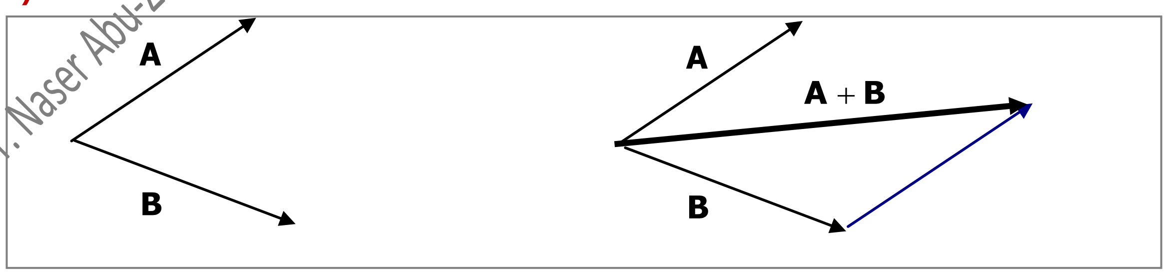
Scalar: A quantity that does not posses direction. Real or complex. (Temperature $T = 20^o$.

Vector addition:

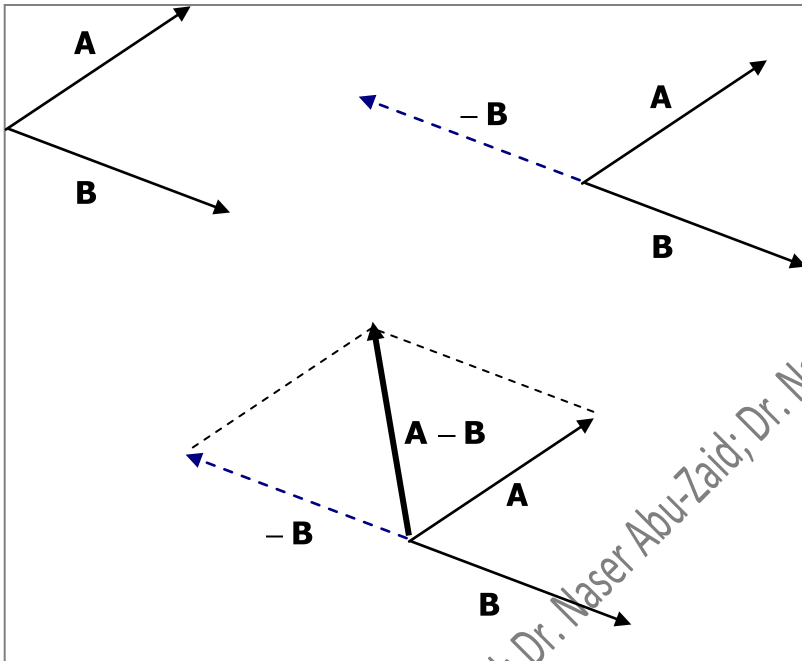
1) Parallelogram:



2) Head to Tail:

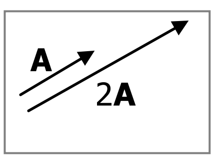


Vector Subtraction:

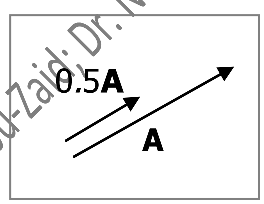


Multiplication by scalar: $\mathbf{B} = k\,\mathbf{A}$

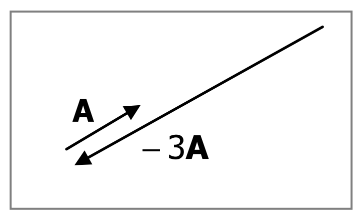
$\mathbf{B} = 2\mathbf{A}$



$\mathbf{B} = 0.5\mathbf{A}$



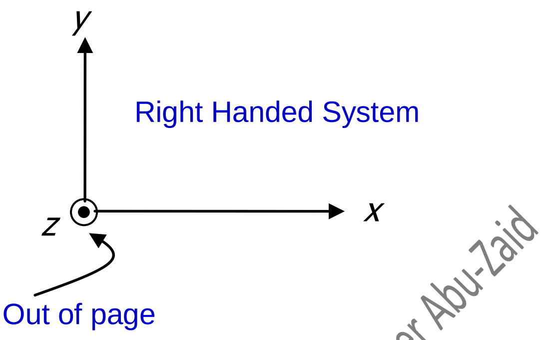
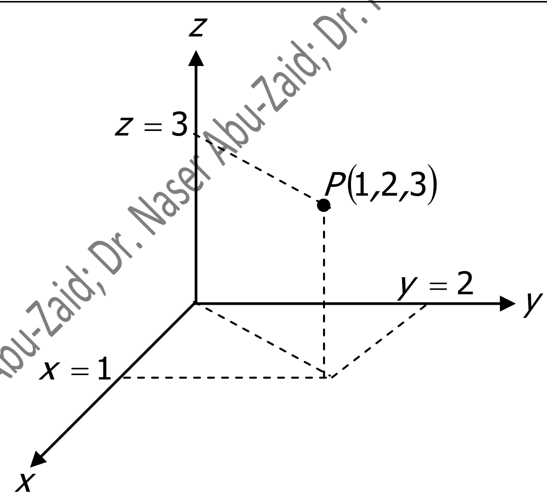
$\mathbf{B} = -3\mathbf{A}$

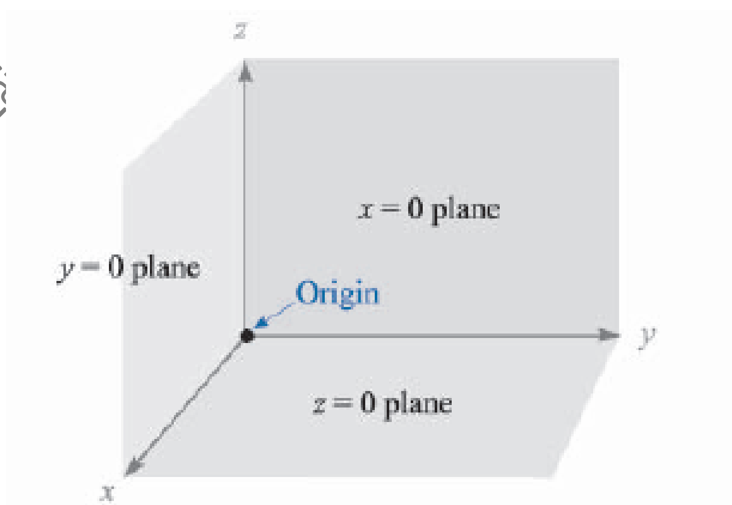


$$(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

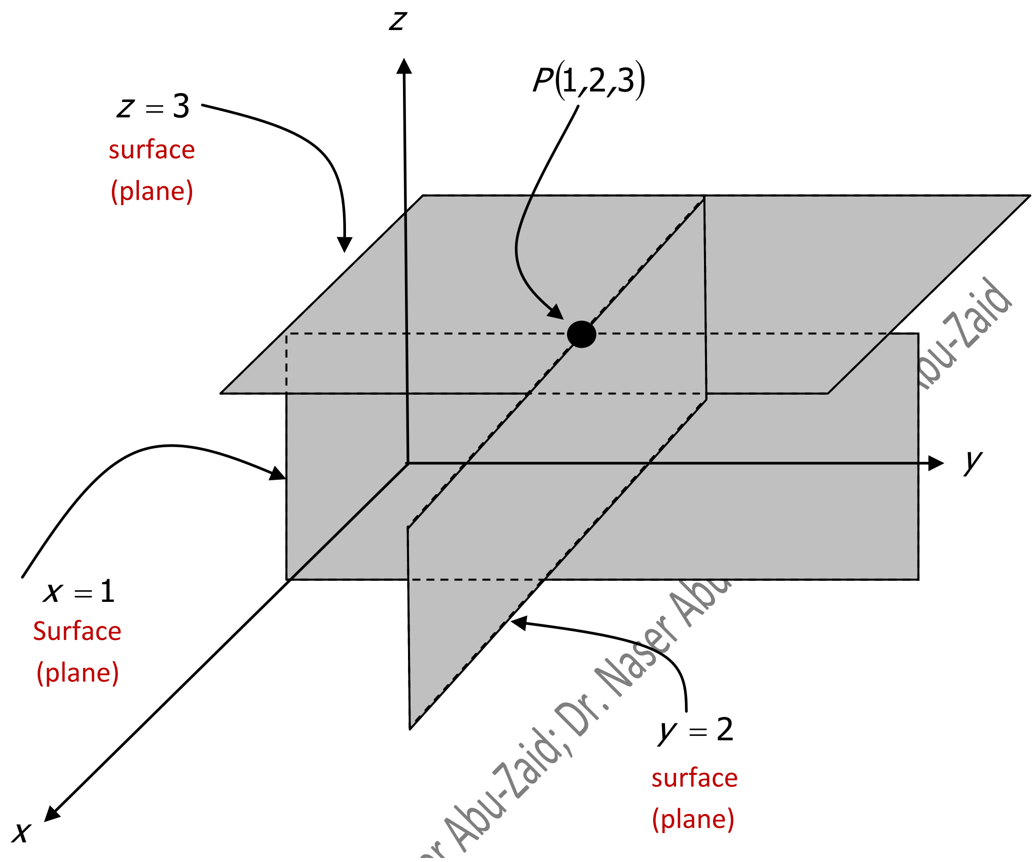
- Commulative law: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Associative law: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- Equal vectors: $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = \mathbf{0}$ (Both have same length and direction)
- Add or subtract vector fields which are defined at the same point.
- If non vector fields are considered then vectors are added or subtracted which are not defined at same point (By shifting them)

THE RECTANGULAR COORDINATE SYSTEM

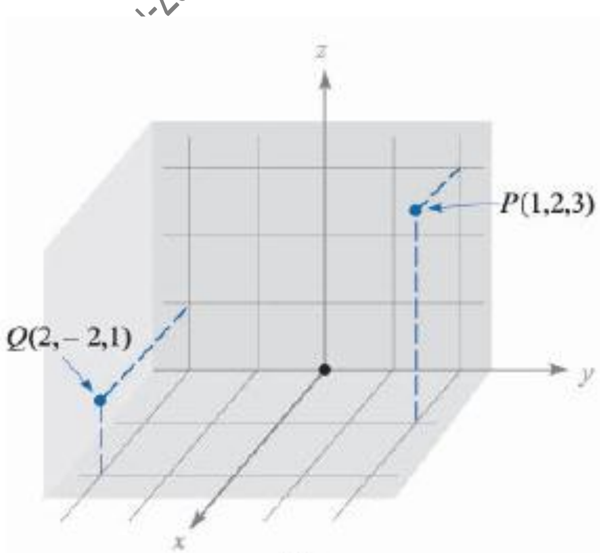
x, y, z are coordinate variables (axis) which are mutually perpendicular.	
A point is located by its x, y and z coordinates, or as the intersection of three constant surfaces (planes in this case)	



(a)

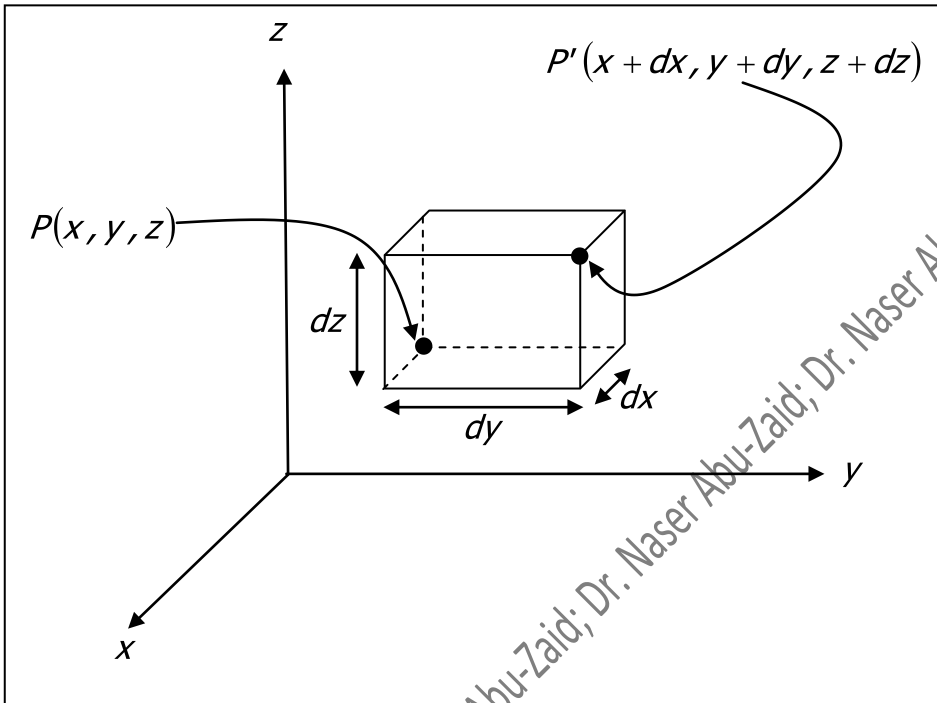


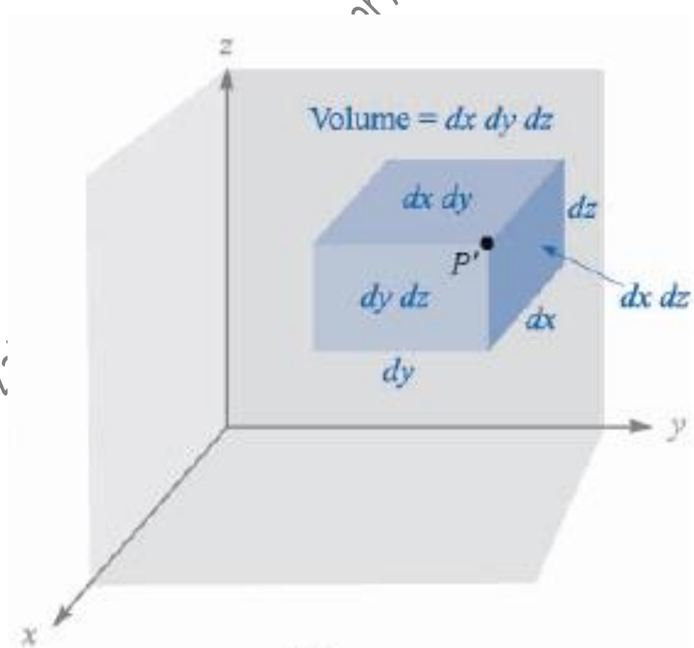
Three mutually perpendicular surfaces intersect at a common point



(b)

Increasing each coordinate variable by a differential amount dx, dy, and dz, one obtains a parallelepiped.





(c)

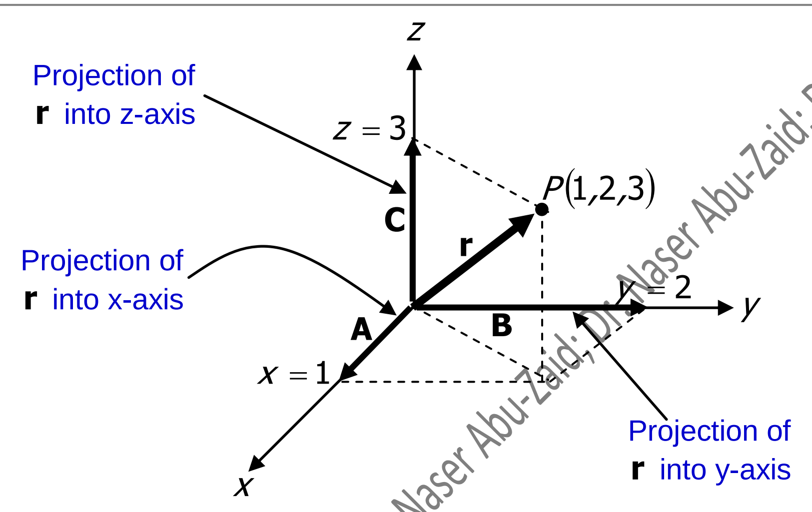
Differential volume: $dv = dxdydz$

Differential Surfaces: Six planes with dierential areas $ds = dxdy$; $ds = dzdy$; $ds = dxdz$

Differential length: from P to P' $dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

VECTOR COMPONENTS AND UNIT VECTORS

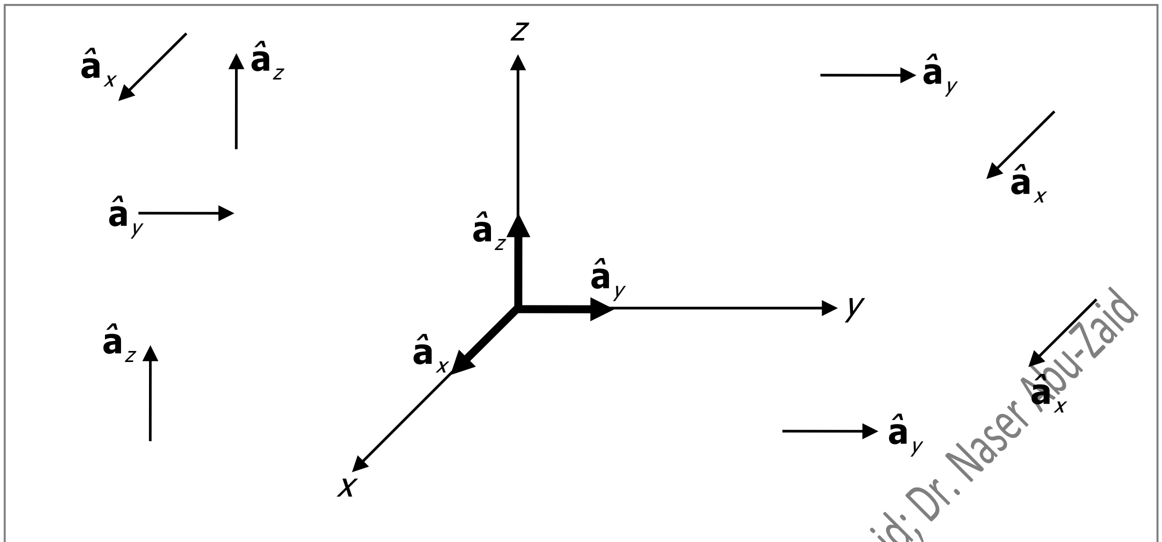
A general vector $\mathbf{r}$ may be written as the sum of three vectors;



$$\mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

$\mathbf{A}, \mathbf{B},$ and $\mathbf{C}$ are ***vector components*** with constant directions.

Unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$ directed along x, y, and z respectively with unity length and no dimensions.



So, the vector $\mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C}$ may be written in terms of unit vectors as:

$$\mathbf{r} = \underbrace{\mathbf{A} + \mathbf{B} + \mathbf{C}}_{\substack{\text{vector components} \\ \mathbf{A},\mathbf{B},\mathbf{C}}} = \underbrace{A\hat{\mathbf{a}}_x + B\hat{\mathbf{a}}_y + C\hat{\mathbf{a}}_z}_{\substack{\text{scalar components} \\ A,B,C}}$$

Where:

A is the directed length or signed magnitude of $\mathbf{A}$.

B is the directed length or signed magnitude of $\mathbf{B}$.

C is the directed length or signed magnitude of $\mathbf{C}$.

As a simple exercise, let $\mathbf{r}_p$ ***(Position vector)*** point from origin (0,0,0) to P(1,2,3), then

$$\mathbf{r}_P = 1\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$$

Scalar components of $\mathbf{r}_P$ are:

$$r_{Px} = A = 1\ ;\ r_{Py} = B = 2;\ r_{Pz} = C = 3.$$

Vector components of $\mathbf{r}_P$ are:

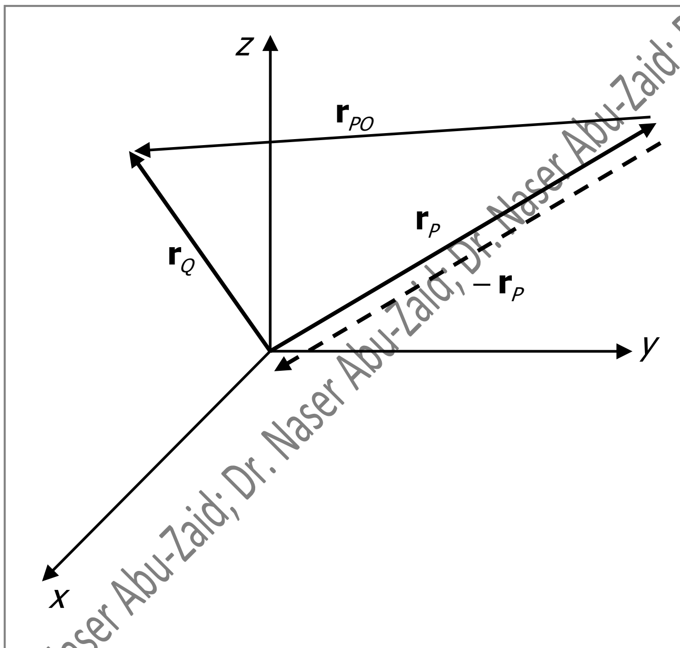
$$\mathbf{r}_{Px} = \mathbf{A} = 1\hat{\mathbf{a}}_x \; ; \; \mathbf{r}_{Py} = \mathbf{B} = 2\hat{\mathbf{a}}_y \; ; \; \mathbf{r}_{Pz} = \mathbf{C} = 3\hat{\mathbf{a}}_z .$$

If Q(2,-2,1) then

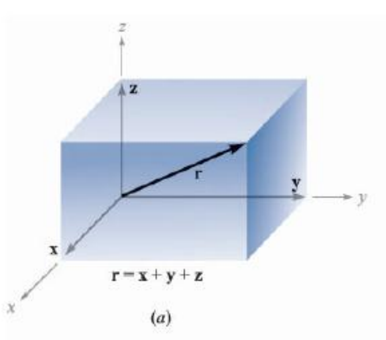
$$\mathbf{r}_Q = 2\hat{\mathbf{a}}_x - 2\hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z$$

And the vector directed from P to Q, $\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P$ *(displacement vector)* which is given by

$$\mathbf{r}_{PQ} = (2-1)\hat{\mathbf{a}}_x + (-2-2)\hat{\mathbf{a}}_y + (1-3)\hat{\mathbf{a}}_z = \hat{\mathbf{a}}_x - 4\hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z$$



The vector $\mathbf{r}_P$ is termed ***position vector*** which is directed from the origin toward the point in quesion.



Other types of vectors (**vector fields** such as Force vector) are denoted:

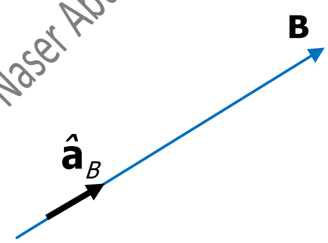
$$\mathbf{F} = F_x\,\hat{\mathbf{a}}_x + F_y\,\hat{\mathbf{a}}_y + F_z\,\hat{\mathbf{a}}_z$$

Where F_x, F_y, F_z are scalar components, and $F_x\,\hat{\mathbf{a}}_x, F_y\,\hat{\mathbf{a}}_y, F_z\,\hat{\mathbf{a}}_z$ are the vector components.

The magnitude of a vector $\mathbf{B} = B_x\,\hat{\mathbf{a}}_x + B_y\,\hat{\mathbf{a}}_y + B_z\,\hat{\mathbf{a}}_z$ is;

$$B = |\mathbf{B}| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}$$

A ***unit vector*** in the direction of $\mathbf{B}$ is;



$$\hat{\mathbf{a}}_B = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{B_x\,\hat{\mathbf{a}}_x + B_y\,\hat{\mathbf{a}}_y + B_z\,\hat{\mathbf{a}}_z}{\sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}}$$

Let $\mathbf{B} = B_x\,\hat{\mathbf{a}}_x + B_y\,\hat{\mathbf{a}}_y + B_z\,\hat{\mathbf{a}}_z$ and $\mathbf{A} = A_x\,\hat{\mathbf{a}}_x + A_y\,\hat{\mathbf{a}}_y + A_z\,\hat{\mathbf{a}}_z$, then

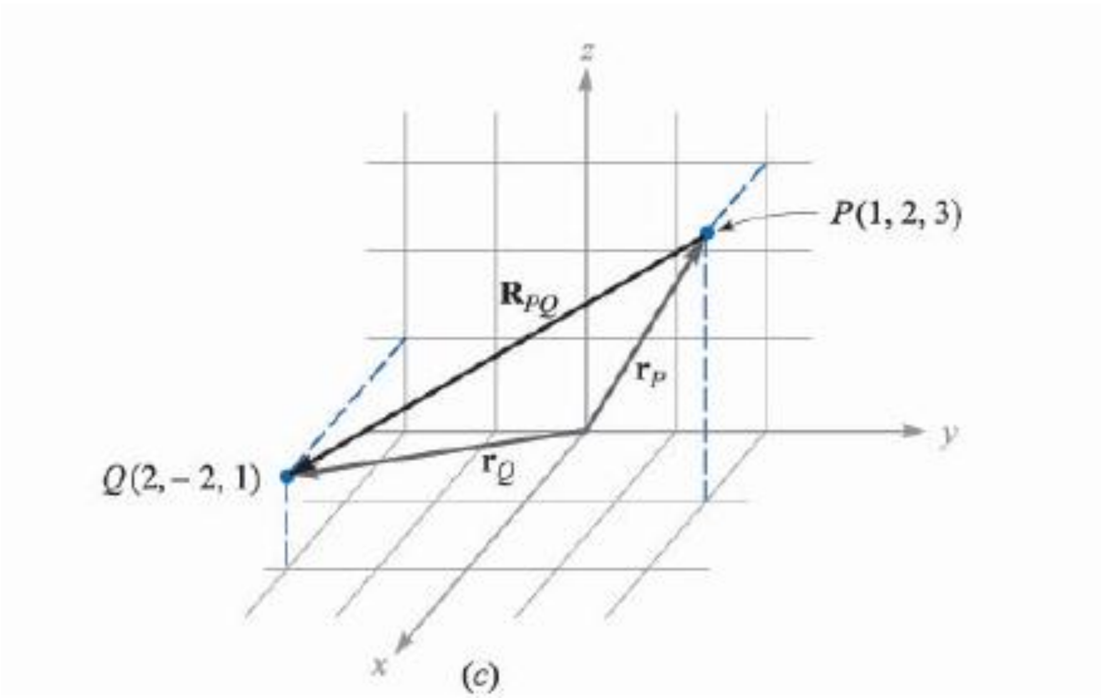
$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{a}}_x + (A_y + B_y)\hat{\mathbf{a}}_y + (A_z + B_z)\hat{\mathbf{a}}_z$$

$$\mathbf{A} - \mathbf{B} = (A_x - B_x)\hat{\mathbf{a}}_x + (A_y - B_y)\hat{\mathbf{a}}_y + (A_z - B_z)\hat{\mathbf{a}}_z$$

Ex: Specify the unit vector extending from the origin toward the point G(2,-2,-1).

Ex: Given M(-1,2,1), N(3,-3,0) and P(-2,-3,-4) Find:

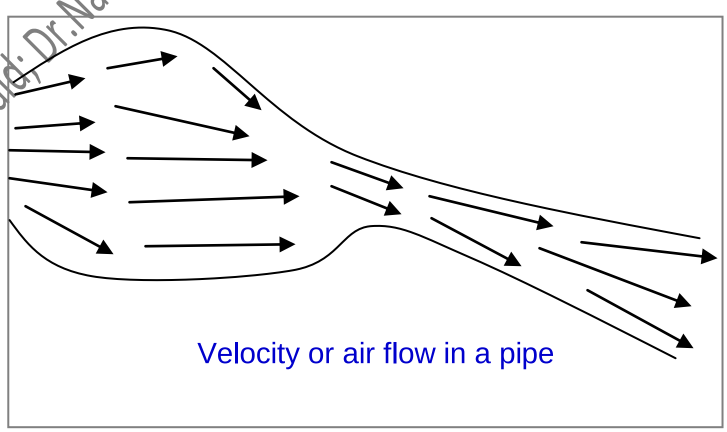
(a) $\mathbf{R}_{MN}$
(b) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$
(c) $|\mathbf{r}_M|$
(d) $\hat{\mathbf{a}}_{MP}$
(e) $|2\mathbf{r}_P - 3\mathbf{r}_N|$



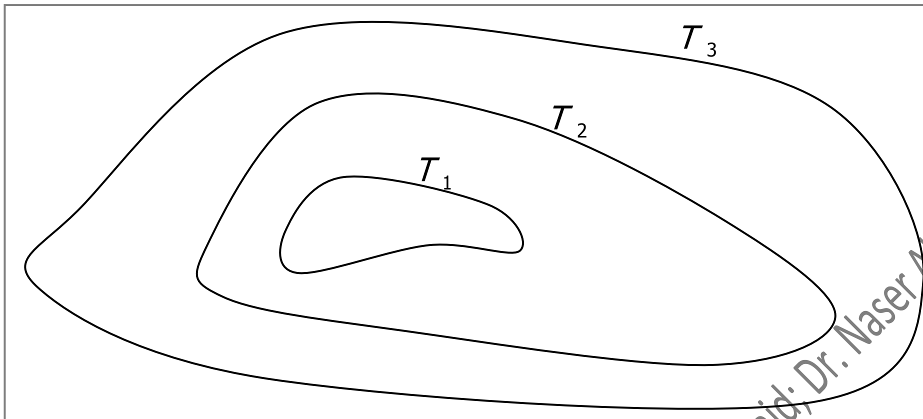
THE VECTOR FIELD AND SCALAR FIELD

Vector Field: vector function of a position vector $\mathbf{r}$. It has a magnitude and direction at each point in space.

$$\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\hat{\mathbf{a}}_x + v_y(\mathbf{r})\hat{\mathbf{a}}_y + v_z(\mathbf{r})\hat{\mathbf{a}}_z$$
$$= v_x(x,y,z)\hat{\mathbf{a}}_x + v_y(x,y,z)\hat{\mathbf{a}}_y + v_z(x,y,z)\hat{\mathbf{a}}_z$$



Scalar field: A scalar function of a position vector $\mathbf{r}$. Temperature is an example $[T(\mathbf{r}) = T(x, y, z)]$ which has a scalar value at each point in space.



Ex: A vector field is expressed as

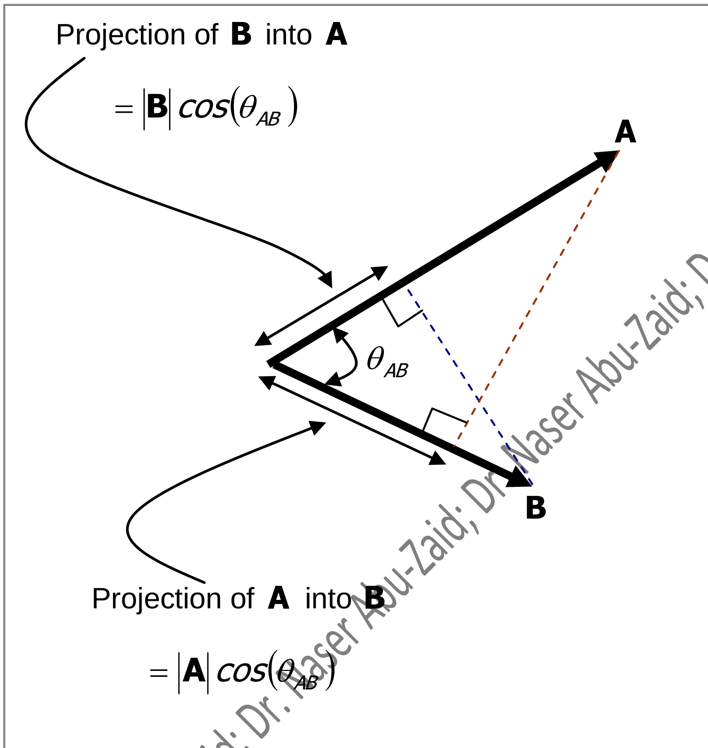
$$\mathbf{S} = \frac{125}{(x-1)^2 + (y-2)^2 + (z+1)^2}\left[(x-1)\hat{\mathbf{a}}_x + (y-2)\hat{\mathbf{a}}_y + (z+1)\hat{\mathbf{a}}_z\right]$$

(a) Is this a scalar or vector field?
(b) Evaluate $\mathbf{S}$ @ $P(2,4,3)$.
(c) Determine a unit vector that gives the direction of $\mathbf{S}$ @ $P(2,4,3)$.
(d) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$.

THE DOT PRODUCT

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta_{AB})$$

Which results in a scalar value, and θ_{AB} is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$.



$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$ since $|\mathbf{A}||\mathbf{B}| \cos(\theta_{AB}) = |\mathbf{B}||\mathbf{A}| \cos(\theta_{AB})$

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_x = |\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_x| \cos(0) = (1)(1)(1) = 1$$

$$\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_y = |\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_y| \cos(0) = (1)(1)(1) = 1$$

$$\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_z| \cos(0) = (1)(1)(1) = 1$$

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_y = |\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_y| \cos(90^o) = (1)(1)(0) = 0 = \hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_x$$

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_z| \cos(90^o) = (1)(1)(0) = 0 = \hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_x$$

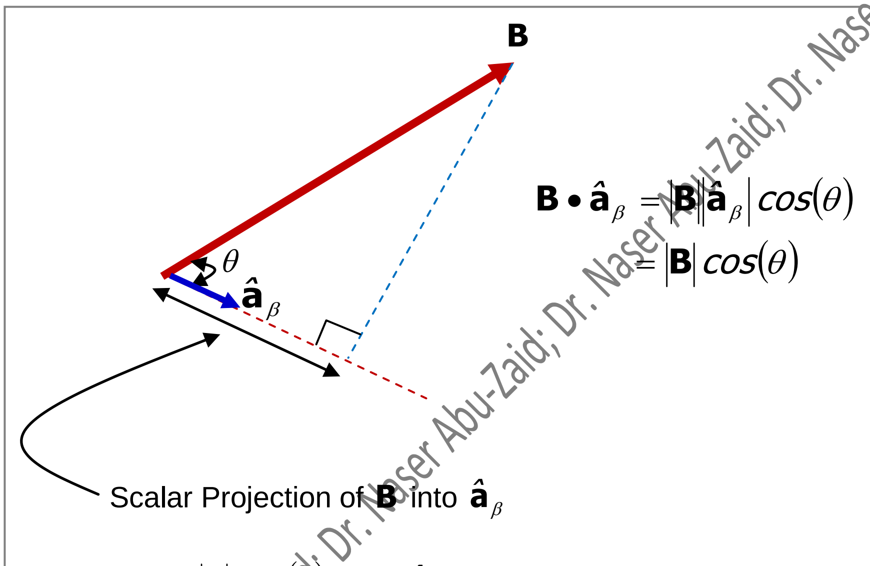
$$\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_z| \cos(90^o) = (1)(1)(0) = 0 = \hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_y$$

Let $\mathbf{B} = B_x\,\hat{\mathbf{a}}_x + B_y\,\hat{\mathbf{a}}_y + B_z\,\hat{\mathbf{a}}_z$ and $\mathbf{A} = A_x\,\hat{\mathbf{a}}_x + A_y\,\hat{\mathbf{a}}_y + A_z\,\hat{\mathbf{a}}_z$, then

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

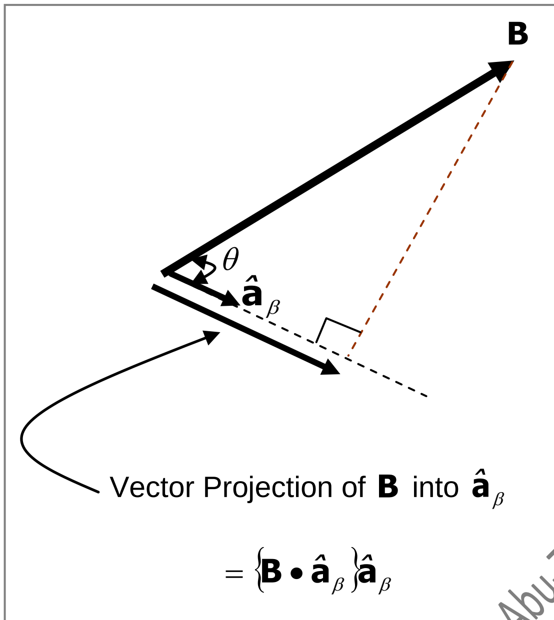
$$\mathbf{A} \bullet \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = |\mathbf{A}|^2 = A^2 \Rightarrow A = \sqrt{\mathbf{A} \bullet \mathbf{A}}$$

The scalar component of $\mathbf{B}$ in the direction of an arbitrary unit vector $\hat{\mathbf{a}}_\beta$ is given by $\mathbf{B} \bullet \hat{\mathbf{a}}_\beta$



$= |\mathbf{B}|\,cos(\theta) = \mathbf{B} \bullet \hat{\mathbf{a}}_\beta$

The vector component of $\mathbf{B}$ in the direction of an arbitrary unit vector $\hat{\mathbf{a}}_\beta$ is given by $(\mathbf{B} \bullet \hat{\mathbf{a}}_\beta)\hat{\mathbf{a}}_\beta$.



Distributive property: $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

Ex: Given $\mathbf{E} = y\,\hat{\mathbf{a}}_x - 2.5x\,\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$ and Q(4,5,2) Find:

(a) $\mathbf{E}$ @ Q.

(b) The scalar component of $\mathbf{E}$ @ Q in the direction of $\hat{\mathbf{a}}_n = \frac{1}{3}(2\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z)$.

(c) The vector component of $\mathbf{E}$ @ Q in the direction of $\hat{\mathbf{a}}_n = \frac{1}{3}(2\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y - 2\hat{\mathbf{a}}_z)$.

(d) The angle θ_{Ea} between $\mathbf{E}(\mathbf{r}_Q)$ and $\hat{\mathbf{a}}_n$.

THE CROSS PRODUCT

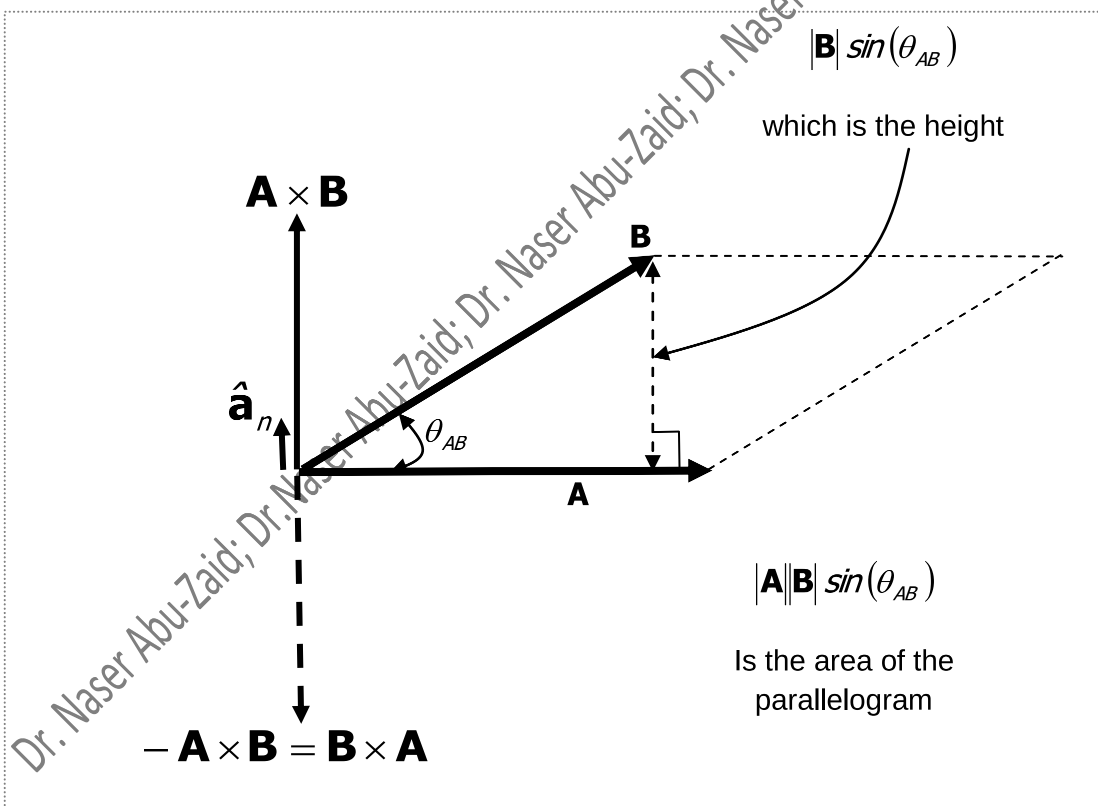
$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\sin(\theta_{AB})\hat{\mathbf{a}}_n$$ results in a vector

$$|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\,\sin(\theta_{AB})$$

$$\textit{Direction of } \mathbf{A}\times\mathbf{B} \Rightarrow \hat{\mathbf{a}}_n$$

$\hat{\mathbf{a}}_n$ is a unit vector normal to the plane containing $\mathbf{A}$ and $\mathbf{B}$. Since there are two possible $\hat{\mathbf{a}}_n's$, we use the ***Right Hand Rule (RHR)*** to determine the direction of $\mathbf{A}\times\mathbf{B}$.

Cross product clearly results in a vector, and θ_{AB} is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$.

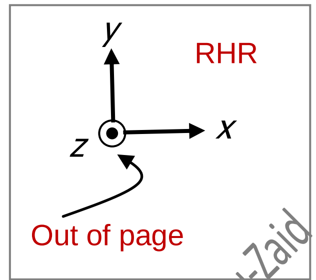


Properties:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}\neq\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$



$$\hat{\mathbf{a}}_x\times\hat{\mathbf{a}}_x=|\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_x|\,sin(0)\hat{\mathbf{a}}_n=0$$

$$\hat{\mathbf{a}}_y\times\hat{\mathbf{a}}_y=|\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_y|\,sin(0)\hat{\mathbf{a}}_n=0$$

$$\hat{\mathbf{a}}_z\times\hat{\mathbf{a}}_z=|\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_z|\,sin(0)\hat{\mathbf{a}}_n=0$$

$$\hat{\mathbf{a}}_x\bullet\hat{\mathbf{a}}_y=|\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_y|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=\hat{\mathbf{a}}_z$$

$$\hat{\mathbf{a}}_x\bullet\hat{\mathbf{a}}_z=|\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_z|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=-\hat{\mathbf{a}}_y$$

$$\hat{\mathbf{a}}_y\bullet\hat{\mathbf{a}}_x=|\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_x|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=-\hat{\mathbf{a}}_z$$

$$\hat{\mathbf{a}}_y\bullet\hat{\mathbf{a}}_z=|\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_z|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=\hat{\mathbf{a}}_x$$

$$\hat{\mathbf{a}}_z\bullet\hat{\mathbf{a}}_x=|\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_x|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=\hat{\mathbf{a}}_y$$

$$\hat{\mathbf{a}}_z\bullet\hat{\mathbf{a}}_y=|\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_y|\,sin(90^o)\hat{\mathbf{a}}_n=(1)(1)(1)\hat{\mathbf{a}}_n=-\hat{\mathbf{a}}_x$$

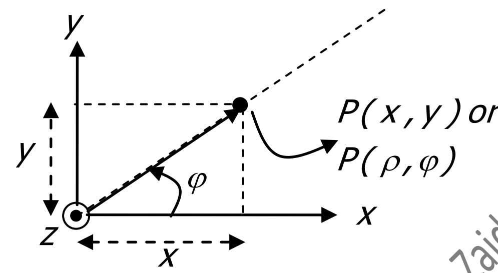
Let $\mathbf{B}=B_x\,\hat{\mathbf{a}}_x+B_y\,\hat{\mathbf{a}}_y+B_z\,\hat{\mathbf{a}}_z$ and $\mathbf{A}=A_x\,\hat{\mathbf{a}}_x+A_y\,\hat{\mathbf{a}}_y+A_z\,\hat{\mathbf{a}}_z$, then

$$\mathbf{A}\times\mathbf{B}=|\mathbf{A}||\mathbf{B}|\,sin(\theta_{AB})\hat{\mathbf{a}}_n=\begin{vmatrix}\hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

$$=(A_yB_z-A_zB_y)\hat{\mathbf{a}}_x-(A_xB_z-A_zB_x)\hat{\mathbf{a}}_y+(A_xB_y-A_yB_x)\hat{\mathbf{a}}_z$$

CIRCULAR CYLINDRICAL COORDINATES

ρ, φ, z are coordinate variables which are mutually perpendicular.



Remember polar coordinates (The 2D version)

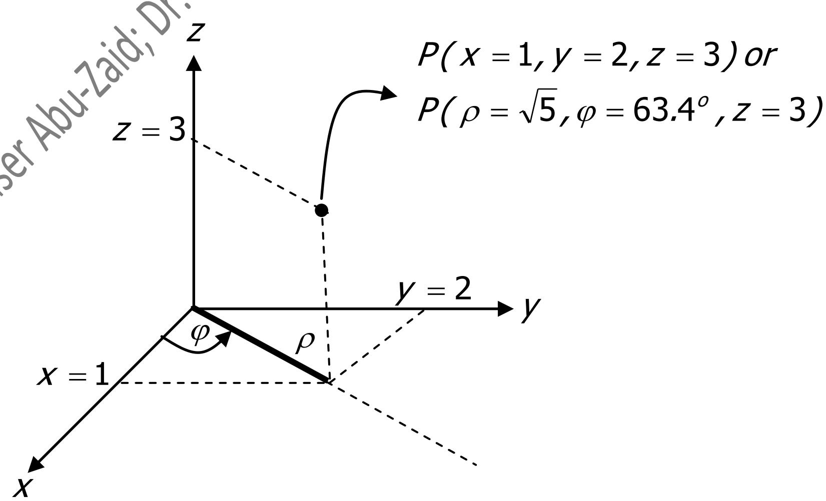
$$x = \rho \cos(\varphi)$$

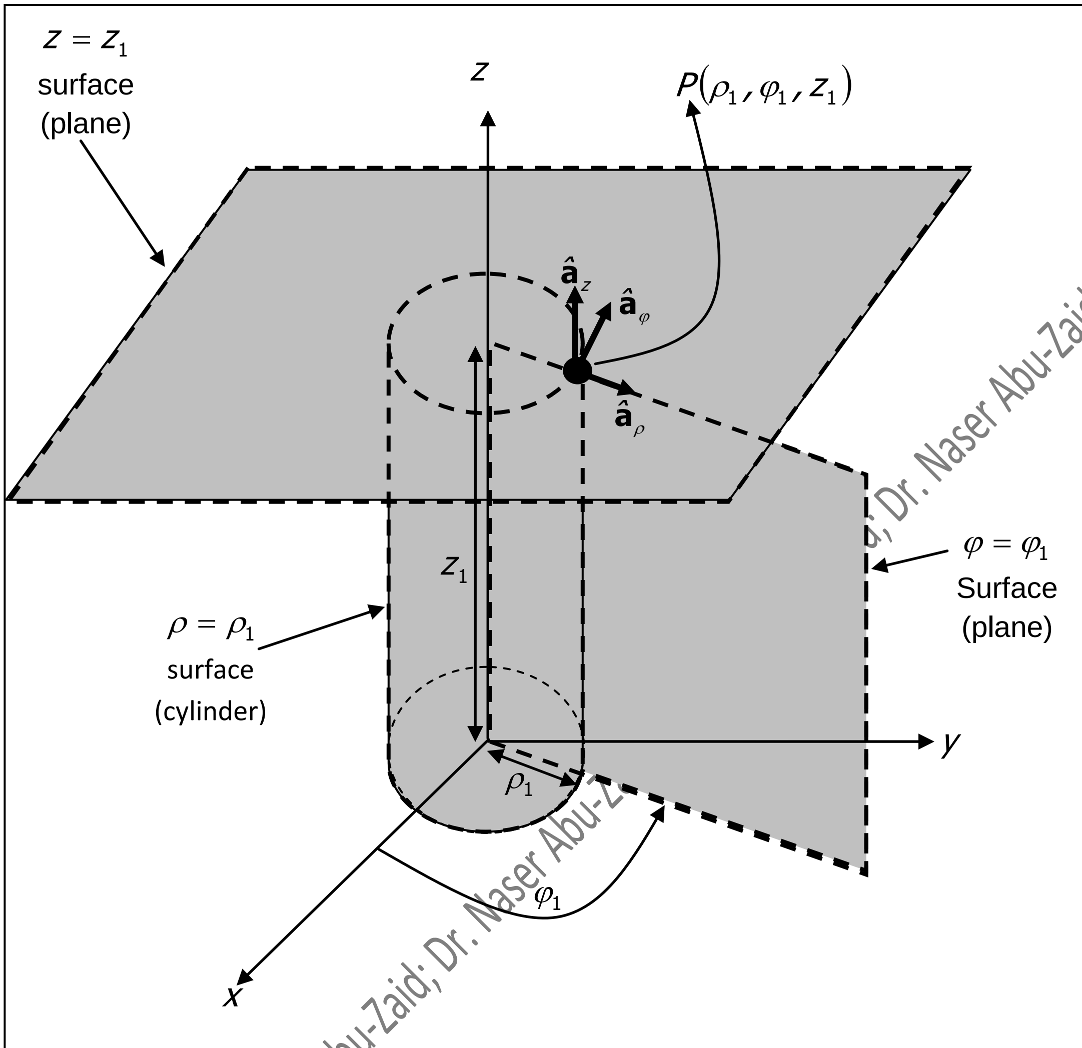
$$y = \rho \sin(\varphi)$$

φ is measured from x-axis toward y-axis.

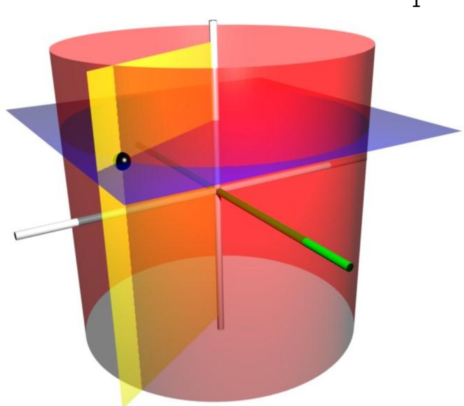
Including the z-coordinate, we obtain the cylindrical coordinates (3D version)

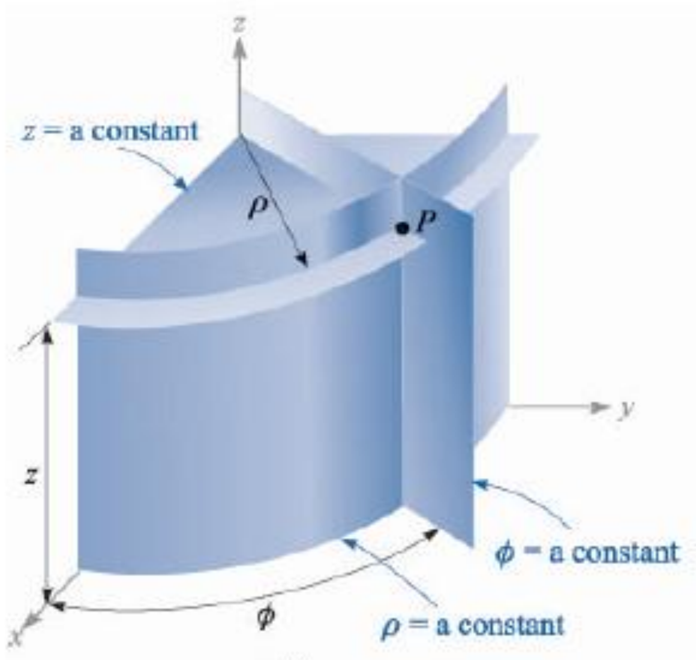
A point is located by its ρ, φ and z coordinates. Or as the intersection of three mutually orthogonal surfaces.





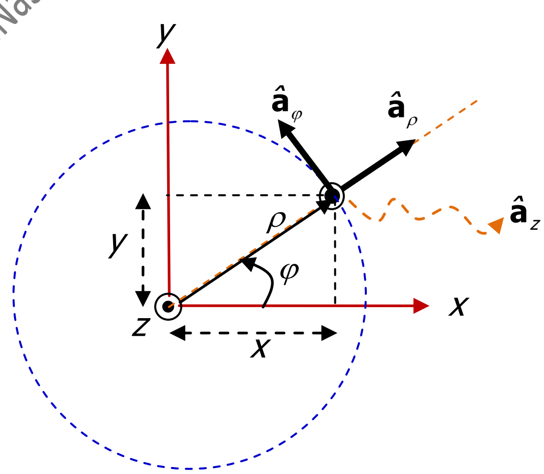
1) Infinitely long cylinder of radius $\rho = \rho_1$.
2) Semi-infinite plane of constant angle $\varphi = \varphi_1$.
3) Infinite plane of constant elevation $z = z_1$.

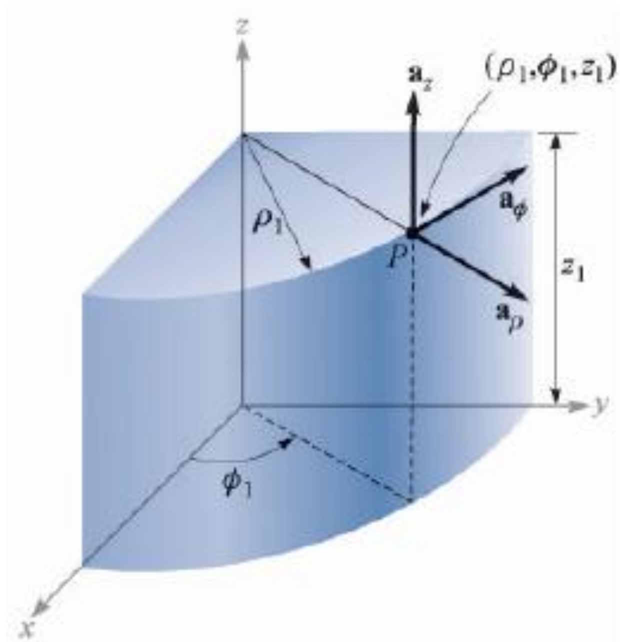




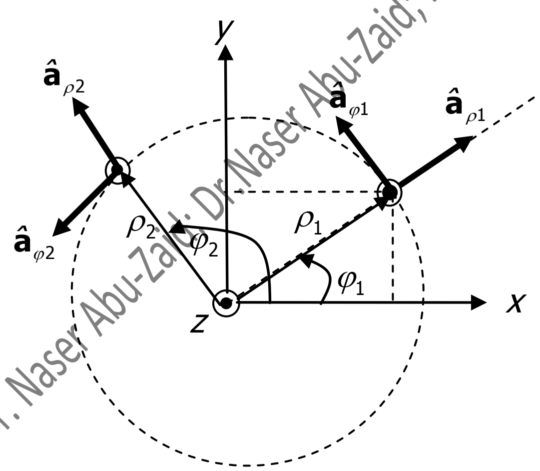
(a)

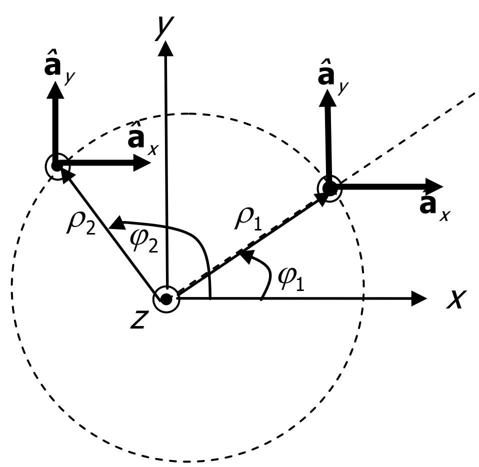
The three unit vectors $\hat{\mathbf{a}}_z$, $\hat{\mathbf{a}}_\varphi$, and $\hat{\mathbf{a}}_\rho$ *are in the direction of increasing variables* and are perpendicular $\perp$ to the surface at which the coordinate variable is constant.





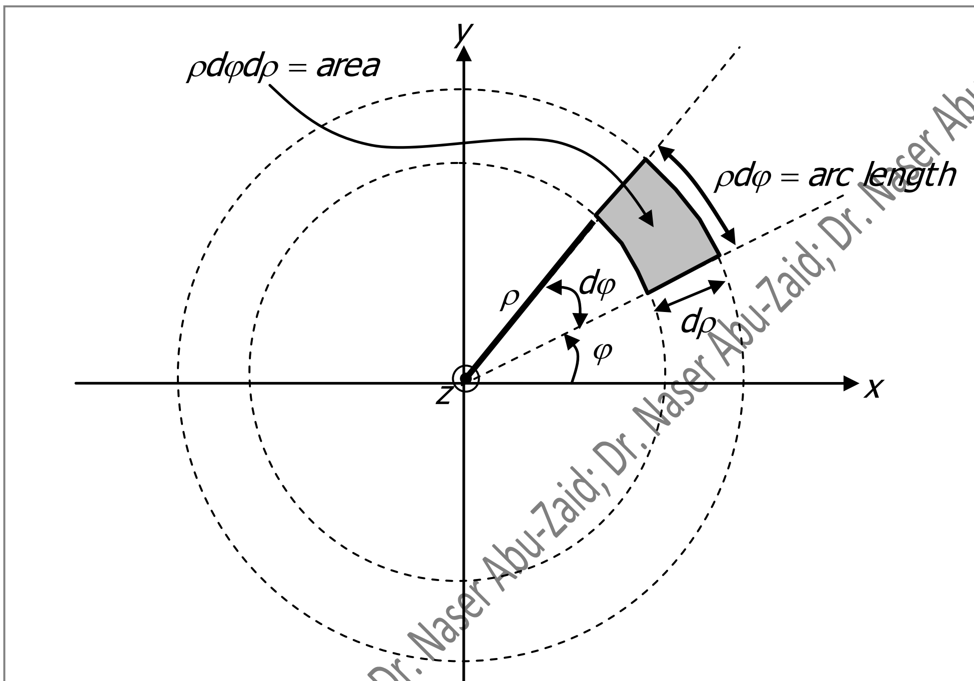
Note that in Cartesian coordinates, unit vectors are not functions of coordinate variables. But in cylindrical coordinates $\hat{\mathbf{a}}_\varphi$, *and* $\hat{\mathbf{a}}_\rho$ *are functions of* φ.





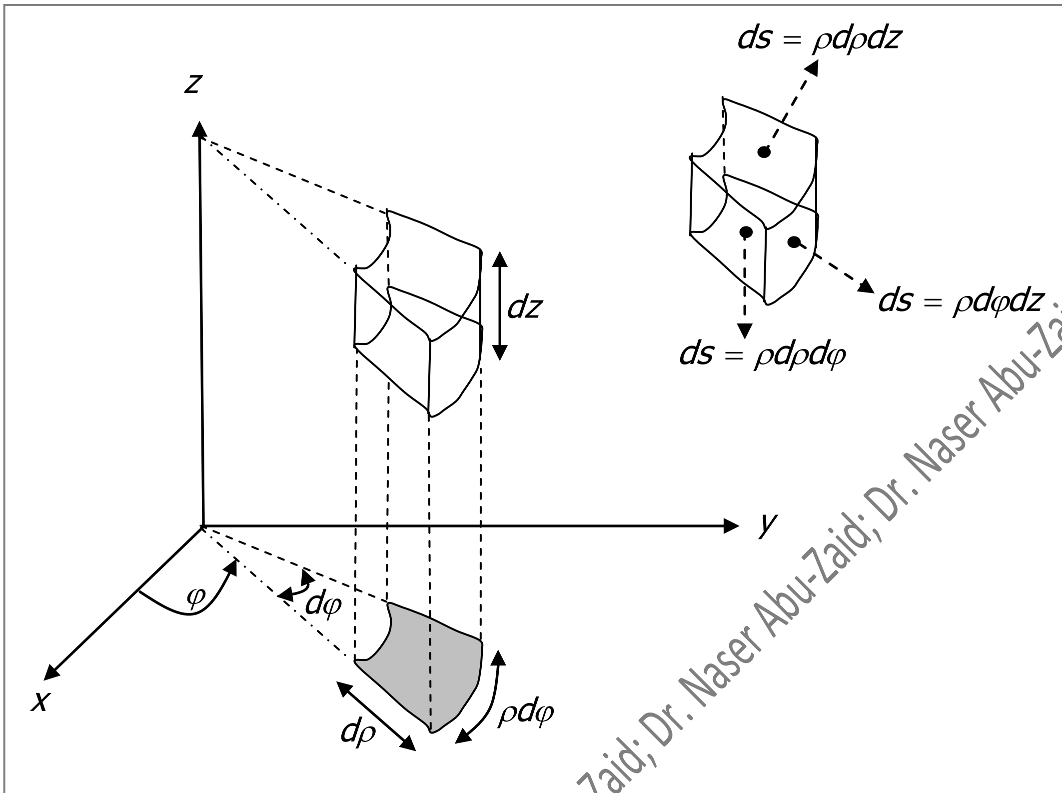
The cylindrical coordinate system is *Right Handed*: $\hat{\mathbf{a}}_\rho \times \hat{\mathbf{a}}_\varphi = \hat{\mathbf{a}}_z$.

Increasing each coordinate variable by a differential amount $d\rho$, $d\varphi$, and dz, one obtains:



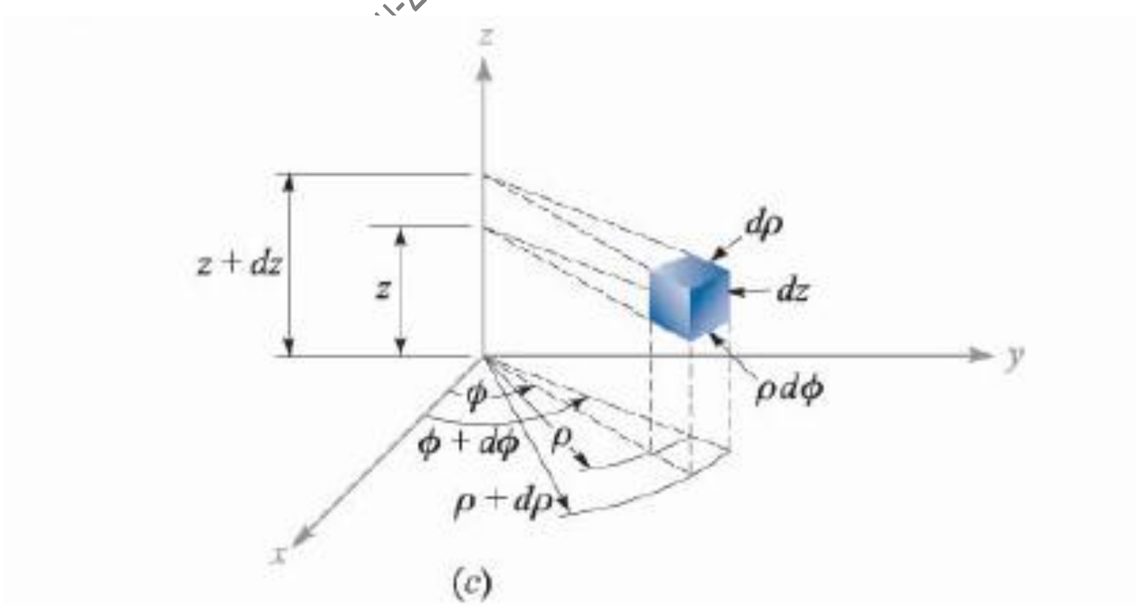
Note that ρ and z are lengths, but φ is an angle which requires a metric coefficient to convert it to length.

$$\text{arc length} = \underbrace{\rho}_{\text{metric coefficient}} \quad d\varphi$$



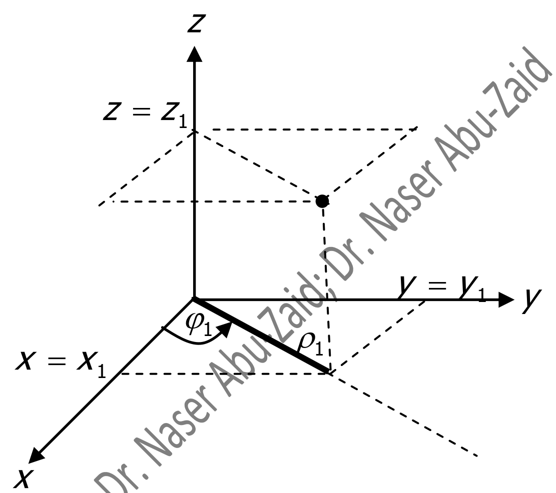
Differential volume: $dv = \rho d\rho d\varphi dz$

Differential Surfaces: Six planes with dierential areas shown in the figure above. (Try it!)



(c)

Transformations between Cylindrical and Cartesian Coordinates



From cylindrical to cart:

$$x = \rho \cos(\varphi)$$

$$y = \rho \sin(\varphi)$$

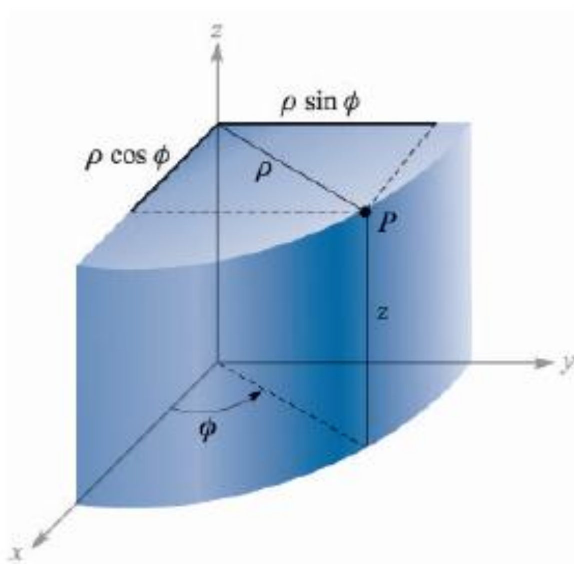
$$z = z$$

From cart. To cyl.:

$$\rho = \sqrt{x^2 + y^2}$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$



Consider a vector in rectangular coordinates;

$$\mathbf{E} = E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z$$

Wishing to write $\mathbf{E}$ in cylindrical coordinates:

$$\mathbf{E} = E_\rho \hat{\mathbf{a}}_\rho + E_\varphi \hat{\mathbf{a}}_\varphi + E_z \hat{\mathbf{a}}_z$$

From the dot product:

$$E_\rho = \mathbf{E} \bullet \hat{\mathbf{a}}_\rho \qquad E_\varphi = \mathbf{E} \bullet \hat{\mathbf{a}}_\varphi \qquad E_z = \mathbf{E} \bullet \hat{\mathbf{a}}_z$$

$$E_\rho = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_\rho$$

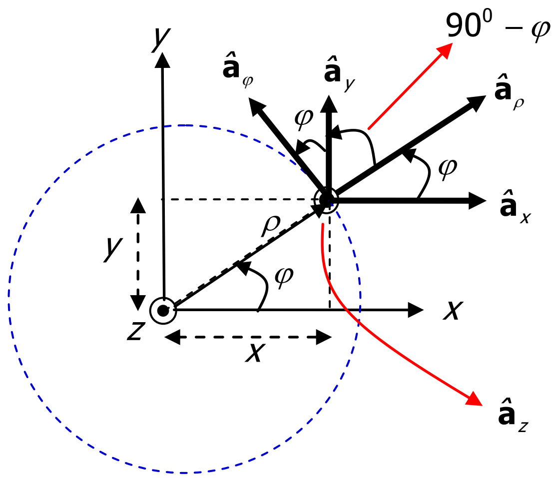
$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\rho}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\rho}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\rho}_{?}$$

$$E_\varphi = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_\varphi$$

$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\varphi}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\varphi}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\varphi}_{?}$$

$$E_z = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_z$$

$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_z}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_z}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_z}_{?}$$



Clearly:

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\rho = |\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_\rho| cos(\varphi) = cos(\varphi)$$

$$\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\rho = |\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_\rho| cos(90^o - \varphi) = sin(\varphi)$$

$$\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\varphi = |\hat{\mathbf{a}}_x||\hat{\mathbf{a}}_\varphi| cos(90^o + \varphi) = -sin(\varphi)$$

$$\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\varphi = |\hat{\mathbf{a}}_y||\hat{\mathbf{a}}_\varphi| cos(\varphi) = cos(\varphi)$$

$$\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\varphi = |\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_\varphi| cos(90^o) = 0$$

$$\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\rho = |\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_\rho| cos(90^o) = 0$$

So:

$$E_\rho = E_x\, cos(\varphi) + E_y\, sin(\varphi)$$

$$E_\varphi = -E_x\, sin(\varphi) + E_y\, cos(\varphi)$$

Or in matrix form

$$\begin{bmatrix} E_\rho \\ E_\varphi \end{bmatrix} = \begin{bmatrix} cos(\varphi) & sin(\varphi) \\ -sin(\varphi) & cos(\varphi) \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

And, the inverse relation is:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} cos(\varphi) & -sin(\varphi) \\ sin(\varphi) & cos(\varphi) \end{bmatrix} \begin{bmatrix} E_\rho \\ E_\varphi \end{bmatrix}$$

Note that the story is not finished here, after *transforming the components*; you should also *transform the coordinate variables*.

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1

Example 1.3

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

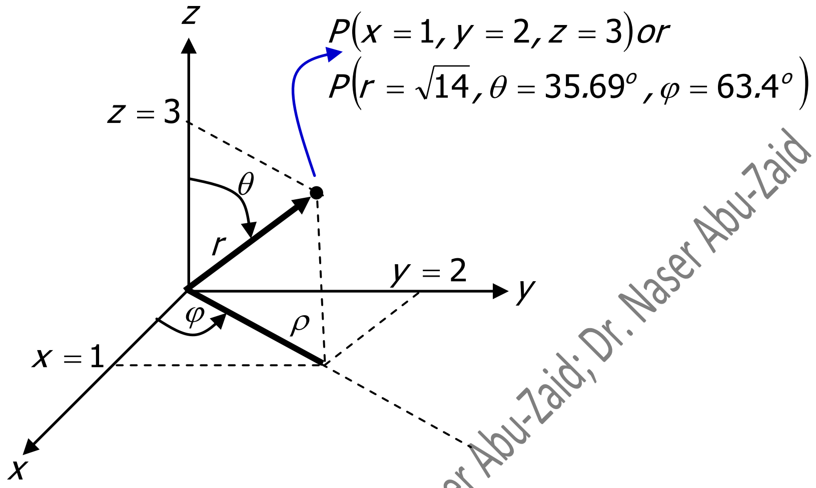
Solution. The new components are

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$
$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$
$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

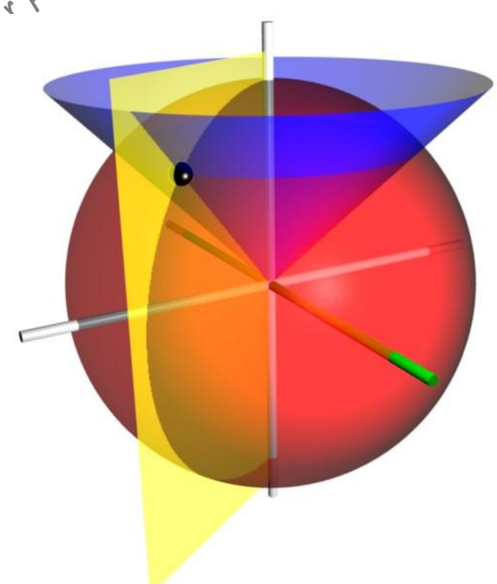
Thus,

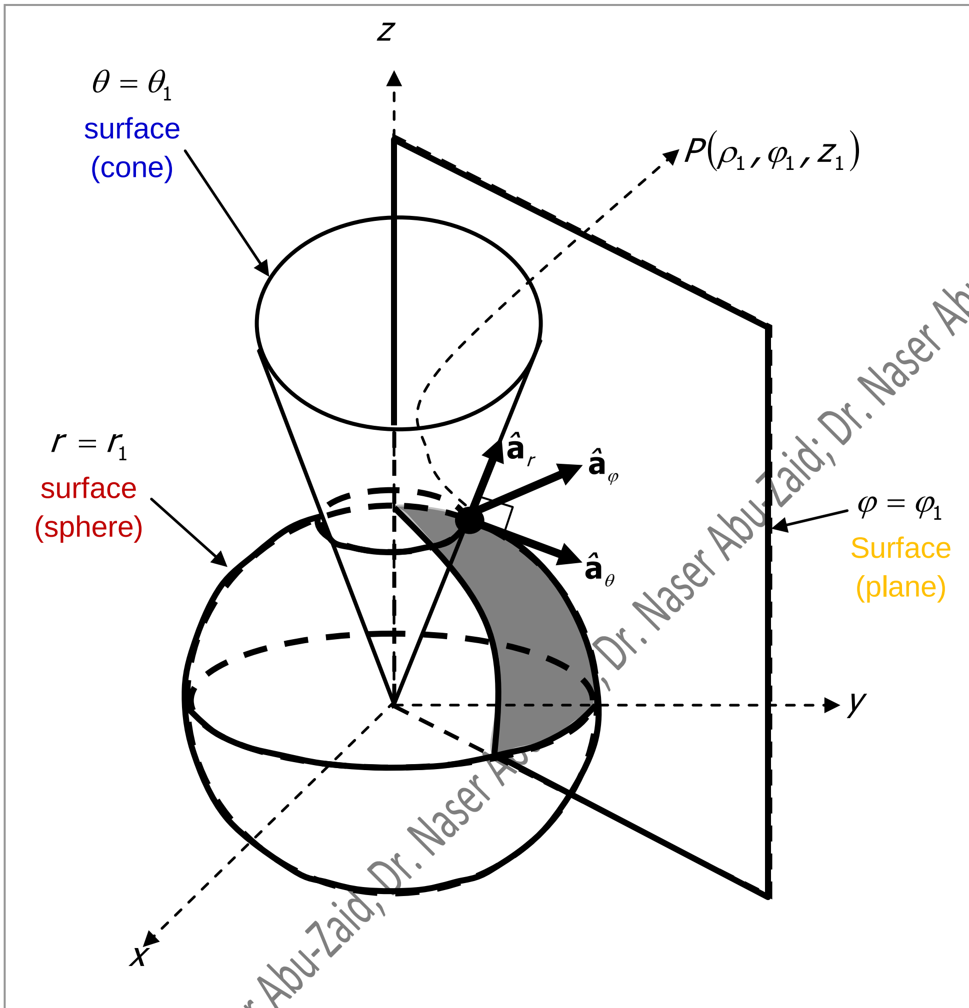
$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

THE SPHERICAL COORDINATE SYSTEM

r, θ, φ are coordinate variables. φ is measured from x-axis toward y-axis, and θ is measured from the z-axis toward the xy plane.	

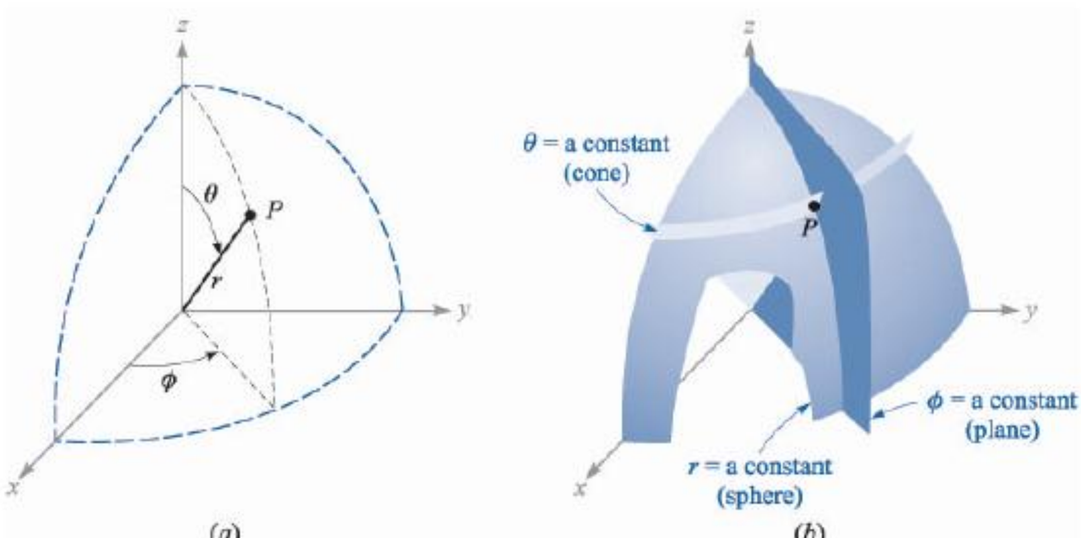
A genral point is loacated by its coordinate variables r, θ, φ, or as the intersection of three mutually perpendicular surfaces.

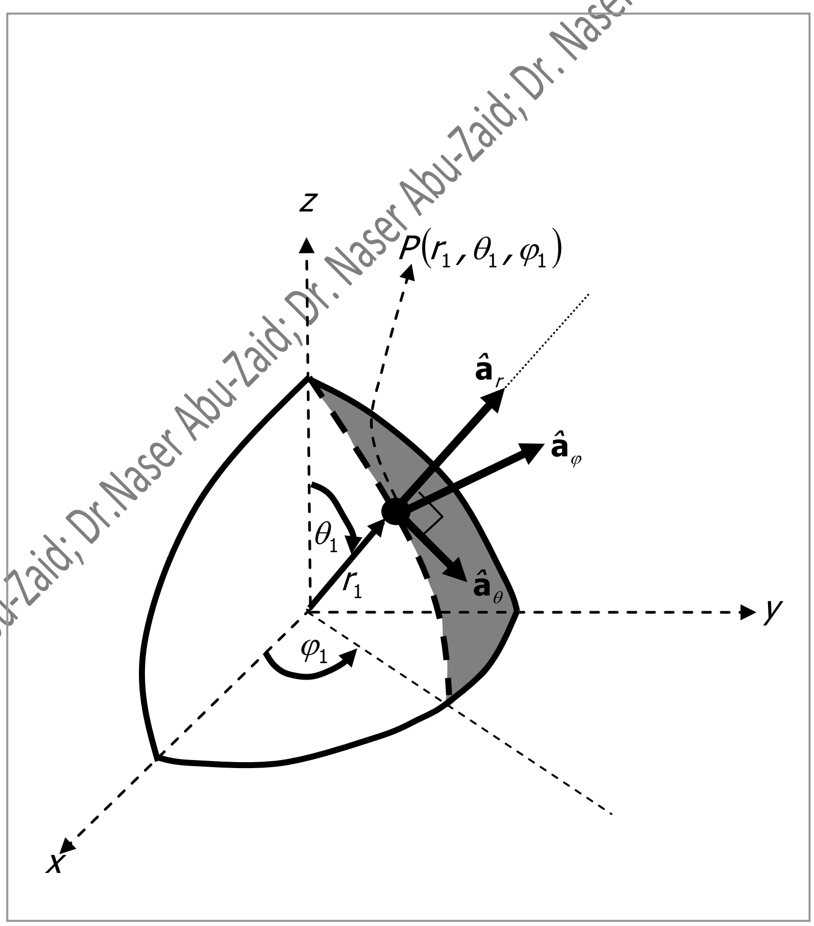


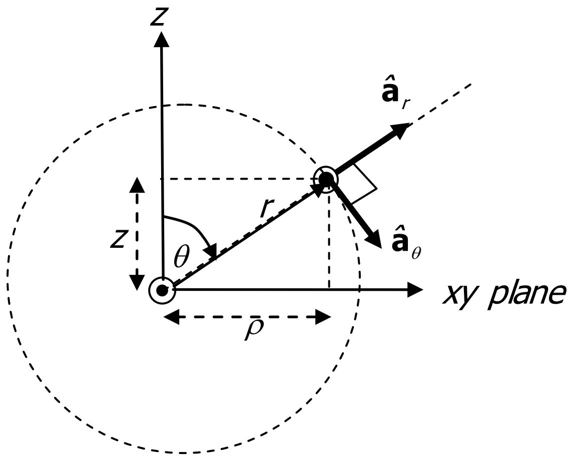


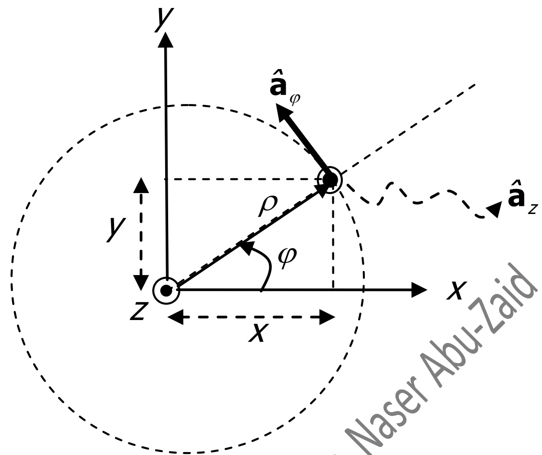
1) Sphere of radius $r = r_1$, centered at the origin.
2) Semi-infinite plane of constant angle $\varphi = \varphi_1$ with it's axis aligned with z-axis.
3) Right angular cone with its apex centered at the origin, and it axis aligned with z-axis, and a cone angle $\theta = \theta_1$.

The three unit vectors $\hat{\mathbf{a}}_r$, $\hat{\mathbf{a}}_\theta$, and $\hat{\mathbf{a}}_\varphi$ are in the direction of increasing variables and are *perpendicular* $\perp$ *to the surface at which the coordinate variable is constant.*

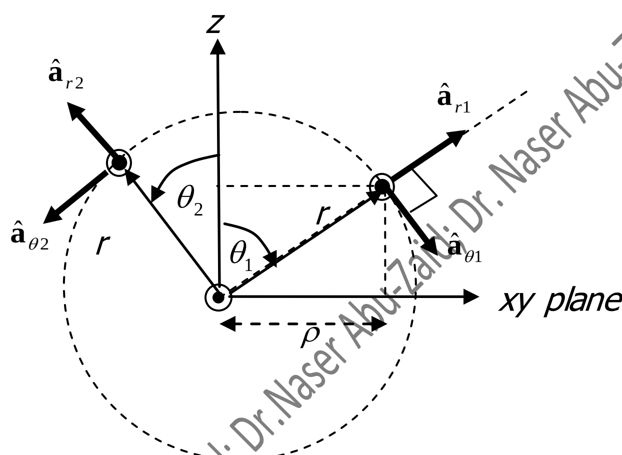

z
θ
P
r
y
φ
x
θ = a constant
(cone)
P
φ = a constant
(plane)
r = a constant
(sphere)
(a)
(b)

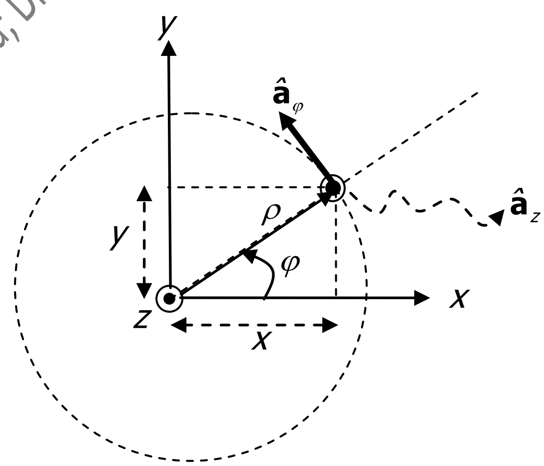

z
$P(r_1, \theta_1, \varphi_1)$
$\hat{\mathbf{a}}_r$
$\hat{\mathbf{a}}_\varphi$
θ_1
r_1
$\hat{\mathbf{a}}_\theta$
y
φ_1
x

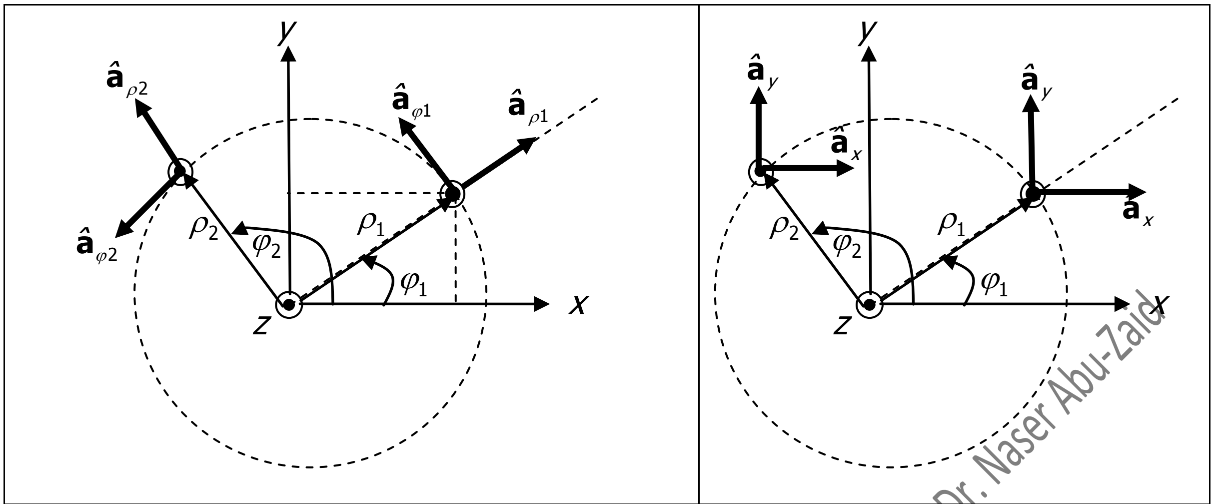




Note that in spherical coordinates, unit vectors are functions of coordinate variables. $\hat{\mathbf{a}}_\varphi$, $\hat{\mathbf{a}}_\theta$ *and* $\hat{\mathbf{a}}_r$ *are functions of* φ *and* θ.



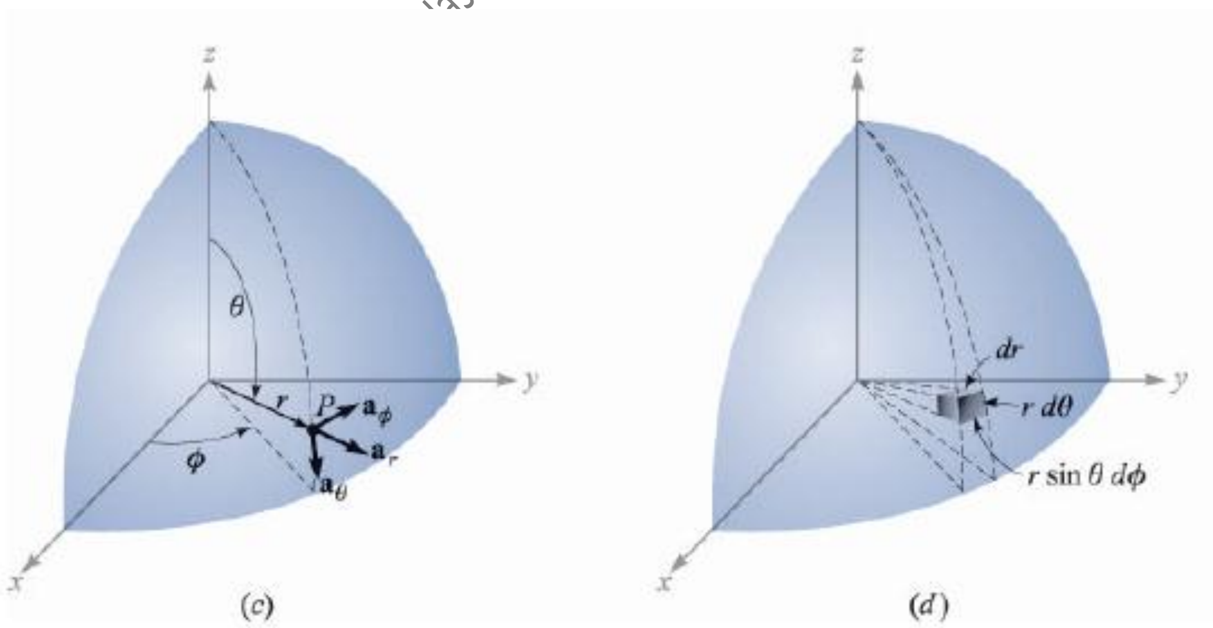




The sphyrical coordinate system is *Right Handed*:

$$\hat{\mathbf{a}}_r \times \hat{\mathbf{a}}_\theta = \hat{\mathbf{a}}_\varphi.$$

Increasing each coordinate variable by a differential amount dr, $d\theta$, and $d\varphi$, one obtains:



Note that r is length, but θ and φ are angles which requires a metric coefficient to convert them to lengths.

$$\text{arc length} = \underbrace{r}_{\substack{metric\\coefficient}} \; d\theta$$

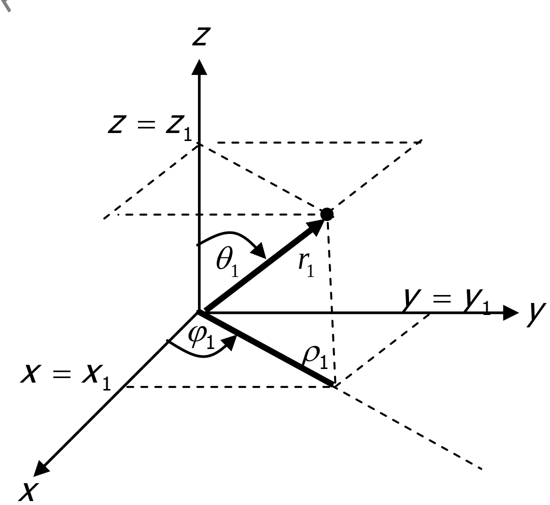
$$\text{arc length} = \underbrace{r\sin\theta}_{\substack{metric\\coefficient}} \; d\varphi$$

Differential volume: $dv = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

Differential Surfaces: Six surfaces with differential areas shown in the figure. (Try itttttttttttt!)

Transformations between Spherical and Cylindrical Coordinates

From spherical to cart:



$x = r\sin(\theta)\cos(\varphi)$

$y = r\sin(\theta)\sin(\varphi)$

$z = r\cos(\theta)$

From cart. To spherical:

$r = \sqrt{x^2 + y^2 + z^2}$; $(r \geq 0)$

$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$; $0 \leq \theta \leq 180^o$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

Consider a vector in rectangular coordinates;

$$\mathbf{E} = E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z$$

Wishing to write **E** in spherical coordinates:

$$\mathbf{E} = E_r \hat{\mathbf{a}}_r + E_\theta \hat{\mathbf{a}}_\theta + E_\varphi \hat{\mathbf{a}}_\varphi$$

From the dot product:

$$E_r = \mathbf{E} \bullet \hat{\mathbf{a}}_r \qquad E_\theta = \mathbf{E} \bullet \hat{\mathbf{a}}_\theta \qquad E_\varphi = \mathbf{E} \bullet \hat{\mathbf{a}}_\varphi$$

$$E_r = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_r$$

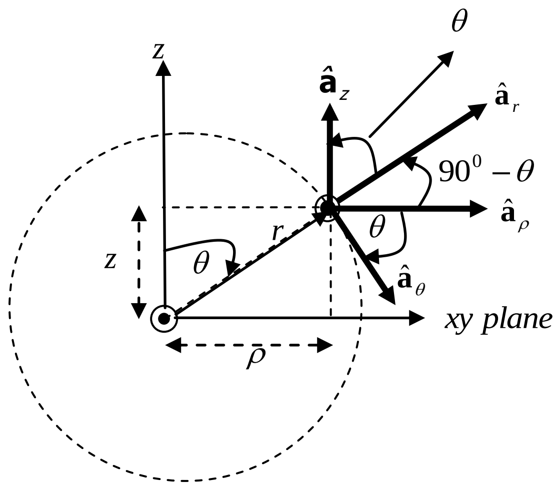
$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_r}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_r}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_r}_{?}$$

$$E_\varphi = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_\varphi$$

$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\varphi}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\varphi}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\varphi}_{?}$$

$$E_\theta = \left(E_x \hat{\mathbf{a}}_x + E_y \hat{\mathbf{a}}_y + E_z \hat{\mathbf{a}}_z\right) \bullet \hat{\mathbf{a}}_\theta$$

$$= E_x \underbrace{\hat{\mathbf{a}}_x \bullet \hat{\mathbf{a}}_\theta}_{?} + E_y \underbrace{\hat{\mathbf{a}}_y \bullet \hat{\mathbf{a}}_\theta}_{?} + E_z \underbrace{\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\theta}_{?}$$



From figure

$$\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_r = |\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_r|\cos(\theta) = \cos(\theta)$$

$$\hat{\mathbf{a}}_\rho \bullet \hat{\mathbf{a}}_r = |\hat{\mathbf{a}}_\rho||\hat{\mathbf{a}}_r|\cos(90^o - \theta) = \sin(\theta)$$

$$\hat{\mathbf{a}}_z \bullet \hat{\mathbf{a}}_\theta = |\hat{\mathbf{a}}_z||\hat{\mathbf{a}}_\theta|\cos(90^o + \theta) = -\sin(\theta)$$

And the rest is left to you as an exercise!

So:

$$E_r = E_x \sin(\theta)\cos(\varphi) + E_y \sin(\theta)\sin(\varphi) + E_z \cos(\theta)$$

Note that, after transforming the components; you should also transform the coordinate variables.

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Example 1.4

We illustrate this transformation procedure by transforming the vector field $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution. We find the three spherical components by dotting **G** with the appropriate unit vectors, and we change variables during the procedure:

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi$$
$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi$$
$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$
$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$
$$= -r\cos\theta\cos\phi$$

Collecting these results, we have

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$