Vector Analysis

Vector algebra:
- Addition; Subtraction;
- Multiplication

Vector Calculus:
- Differentiation; Integration

**Preliminary material**
**(mathematical requirements)**

**Vector**: A quantity with both magnitude and direction. (Force $\mathbf{F} = 10\text{N}$ to the east).

**Scalar**: A quantity that does not possess direction, Real or complex. (Temperature $T = 20^\circ$).

**Vector addition:**

1) **Parallelogram:**

2) **Head to Tail:**
Vector Subtraction:

\[ \mathbf{A} - \mathbf{B} \]

Multiplication by scalar: \( \mathbf{B} = k \mathbf{A} \)

\( \mathbf{B} = 2\mathbf{A} \)

\[ \begin{align*}
\mathbf{A} & \quad \text{2A} \\
\end{align*} \]

\( \mathbf{B} = 0.5\mathbf{A} \)

\[ \begin{align*}
\mathbf{A} & \quad \text{0.5A} \\
\end{align*} \]

\( \mathbf{B} = -3\mathbf{A} \)

\[ \begin{align*}
\mathbf{A} & \quad \text{-3A} \\
\end{align*} \]

\[(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}\]

- Commutative law: \( \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \)
- Associative law: \( (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \)
- Equal vectors: \( \mathbf{A} = \mathbf{B} \) if \( \mathbf{A} - \mathbf{B} = \mathbf{0} \) (Both have same length and direction)
- Add or subtract vector fields which are defined at the same point.
- If non vector fields are considered then vectors are added or subtracted which are not defined at same point (By shifting them)
**THE RECTANGULAR COORDINATE SYSTEM**

$x, y, z$ are coordinate variables (axis) which are mutually perpendicular.

A point is located by its $x, y$ and $z$ coordinates, or as the intersection of three constant surfaces (planes in this case).
Three mutually perpendicular surfaces intersect at a common point.
Increasing each coordinate variable by a differential amount $dx$, $dy$, and $dz$, one obtains a parallelepiped.

Differential volume: $dv = dx\, dy\, dz$
Differential Surfaces: Six planes with differential areas $ds = dxdy$; $ds = dzdy$; $ds = dxdz$

Differential length: from $P$ to $P'$ $dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

**VECTOR COMPONENTS AND UNIT VECTORS**

A general vector $\mathbf{r}$ may be written as the sum of three vectors;

$\mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C}$

$\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are *vector components* with constant directions.

*Unit vectors* $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$ directed along $x$, $y$, and $z$ respectively with unity length and no dimensions.
So, the vector \( \mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C} \) may be written in terms of unit vectors as:

\[
\mathbf{r} = \mathbf{A} + \mathbf{B} + \mathbf{C} = A \hat{\mathbf{a}}_x + B \hat{\mathbf{a}}_y + C \hat{\mathbf{a}}_z
\]

Where:

- \( A \) is the directed length or signed magnitude of \( \mathbf{A} \).
- \( B \) is the directed length or signed magnitude of \( \mathbf{B} \).
- \( C \) is the directed length or signed magnitude of \( \mathbf{C} \).

As a simple exercise, let \( \mathbf{r}_p \) (Position vector) point from origin \((0,0,0)\) to \(P(1,2,3)\), then

\[
\mathbf{r}_p = 1\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z
\]

Scalar components of \( \mathbf{r}_p \) are:

\[
r_{px} = A = 1; \quad r_{py} = B = 2; \quad r_{pz} = C = 3.
\]

Vector components of \( \mathbf{r}_p \) are:
\[ \mathbf{r}_{px} = \mathbf{A} = 1\hat{a}_x; \quad \mathbf{r}_{py} = \mathbf{B} = 2\hat{a}_y; \quad \mathbf{r}_{pz} = \mathbf{C} = 3\hat{a}_z. \]

If \( Q(2, -2, 1) \) then

\[ \mathbf{r}_Q = 2\hat{a}_x - 2\hat{a}_y + \hat{a}_z \]

And the vector directed from \( P \) to \( Q \), \( \mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P \) (displacement vector), which is given by

\[ \mathbf{r}_{PQ} = (2 - 1)\hat{a}_x + (-2 - 2)\hat{a}_y + (1 - 3)\hat{a}_z = \hat{a}_x - 4\hat{a}_y - 2\hat{a}_z \]

The vector \( \mathbf{r}_p \) is termed \textit{position vector} which is directed from the origin toward the point in question.
Other types of vectors (vector fields) such as Force vector are denoted:

\[ \mathbf{F} = F_x \hat{a}_x + F_y \hat{a}_y + F_z \hat{a}_z \]

Where \( F_x, F_y, F_z \) are scalar components, and \( F_x \hat{a}_x, F_y \hat{a}_y, F_z \hat{a}_z \) are the vector components.

The magnitude of a vector \( \mathbf{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \) is:

\[ B = |\mathbf{B}| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2} \]

A unit vector in the direction of \( \mathbf{B} \) is:

\[ \hat{a}_B = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z}{\sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}} \]

Let \( \mathbf{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z \) and \( \mathbf{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \), then

\[ \mathbf{A} + \mathbf{B} = (A_x + B_x) \hat{a}_x + (A_y + B_y) \hat{a}_y + (A_z + B_z) \hat{a}_z \]
\[ \mathbf{A} - \mathbf{B} = (A_x - B_x) \hat{a}_x + (A_y - B_y) \hat{a}_y + (A_z - B_z) \hat{a}_z \]

Ex: Specify the unit vector extending from the origin toward the point G(2, -2, -1).

Ex: Given M(-1, 2, 1), N(3, -3, 0) and P(-2, -3, -4) Find:

(a) \( \mathbf{R}_{MN} \)
(b) \( \mathbf{R}_{MN} + \mathbf{R}_{NP} \)
(c) \( |\mathbf{r}_M| \)
(d) \( \hat{a}_{MP} \)
(e) \( |2\mathbf{r}_P - 3\mathbf{r}_N| \)
THE VECTOR FIELD AND SCALAR FIELD

**Vector Field:** vector function of a position vector \( \mathbf{r} \). It has a magnitude and direction at each point in space.

\[
\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r}) \hat{\mathbf{a}}_x + v_y(\mathbf{r}) \hat{\mathbf{a}}_y + v_z(\mathbf{r}) \hat{\mathbf{a}}_z
\]

\[
= v_x(x, y, z) \hat{\mathbf{a}}_x + v_y(x, y, z) \hat{\mathbf{a}}_y + v_z(x, y, z) \hat{\mathbf{a}}_z
\]

**Velocity or air flow in a pipe**
**Scalar field:** A scalar function of a position vector $\mathbf{r}$. Temperature is an example $[T(\mathbf{r}) = T(x, y, z)]$ which has a scalar value at each point in space.

**Example:** A vector field is expressed as

$$\mathbf{S} = \frac{125}{(x-1)^2 + (y-2)^2 + (z+1)^2} \left[ (x-1)\hat{a}_x + (y-2)\hat{a}_y + (z+1)\hat{a}_z \right]$$

(a) Is this a scalar or vector field?

(b) Evaluate $\mathbf{S} @ P(2,4,3)$.

(c) Determine a unit vector that gives the direction of $\mathbf{S} @ P(2,4,3)$.

(d) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$. 
THE DOT PRODUCT

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos(\theta_{AB}) \]

Which results in a scalar value, and \( \theta_{AB} \) is the smaller angle between \( \mathbf{A} \) and \( \mathbf{B} \).

\[ \text{Projection of } \mathbf{B} \text{ into } \mathbf{A} = |\mathbf{B}| \cos(\theta_{AB}) \]

\[ \text{Projection of } \mathbf{A} \text{ into } \mathbf{B} = |\mathbf{A}| \cos(\theta_{AB}) \]

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \text{ since } |\mathbf{A}| |\mathbf{B}| \cos(\theta_{AB}) = |\mathbf{B}| |\mathbf{A}| \cos(\theta_{AB}) \]

\[ \hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_x = |\hat{\mathbf{a}}_x| |\hat{\mathbf{a}}_x| \cos(0) = (1)(1) = 1 \]

\[ \hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_y = |\hat{\mathbf{a}}_y| |\hat{\mathbf{a}}_y| \cos(0) = (1)(1) = 1 \]

\[ \hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_z| |\hat{\mathbf{a}}_z| \cos(0) = (1)(1) = 1 \]

\[ \hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_y = |\hat{\mathbf{a}}_x| |\hat{\mathbf{a}}_y| \cos(90^\circ) = (1)(1) = 0 = \hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x \]

\[ \hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_x| |\hat{\mathbf{a}}_z| \cos(90^\circ) = (1)(1) = 0 = \hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_x \]

\[ \hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_z = |\hat{\mathbf{a}}_y| |\hat{\mathbf{a}}_z| \cos(90^\circ) = (1)(1) = 0 = \hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_y \]
Let $\mathbf{B} = B_x \hat{a}_x + B_y \hat{a}_y + B_z \hat{a}_z$ and $\mathbf{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$, then

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = |\mathbf{A}|^2 = A^2 \Rightarrow A = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

The scalar component of $\mathbf{B}$ in the direction of an arbitrary unit vector $\hat{a}_\beta$ is given by $\mathbf{B} \cdot \hat{a}_\beta$

$$\mathbf{B} \cdot \hat{a}_\beta = |\mathbf{B}| \hat{a}_\beta \cos(\theta)$$

The vector component of $\mathbf{B}$ in the direction of an arbitrary unit vector $\hat{a}_\beta$ is given by $(\mathbf{B} \cdot \hat{a}_\beta) \hat{a}_\beta$. 
Distributive property: \( A \cdot (B + C) = A \cdot B + A \cdot C \)

**Ex:** Given \( E = y \hat{a}_x - 2.5x \hat{a}_y + 3 \hat{a}_z \) and \( Q(4,5,2) \) Find:

(a) \( E @ Q \).
(b) The scalar component of \( E @ Q \) in the direction of \( \hat{a}_n = \frac{1}{3} (2 \hat{a}_x + \hat{a}_y - 2 \hat{a}_z) \).
(c) The vector component of \( E @ Q \) in the direction of \( \hat{a}_n = \frac{1}{3} (2 \hat{a}_x + \hat{a}_y - 2 \hat{a}_z) \).
(d) The angle \( \theta_{E\hat{a}} \) between \( E(\mathbf{r}_Q) \) and \( \hat{a}_n \).
THE CROSS PRODUCT

\[ \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin(\theta_{AB}) \hat{\mathbf{a}}_n \text{ results in a vector} \]

\[ |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin(\theta_{AB}) \]

Direction of \( \mathbf{A} \times \mathbf{B} \Rightarrow \hat{\mathbf{a}}_n \)

\( \hat{\mathbf{a}}_n \) is a unit vector normal to the plane containing \( \mathbf{A} \) and \( \mathbf{B} \). Since there are two possible \( \hat{\mathbf{a}}_{n'} \)s, we use the Right Hand Rule (RHR) to determine the direction of \( \mathbf{A} \times \mathbf{B} \).

Cross product clearly results in a vector, and \( \theta_{AB} \) is the smaller angle between \( \mathbf{A} \) and \( \mathbf{B} \).

Properties:

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \]

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \]
\( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \)

\[ \mathbf{\hat{a}}_x \times \mathbf{\hat{a}}_x = |\mathbf{\hat{a}}_x| |\mathbf{\hat{a}}_x| \sin(0) \mathbf{\hat{a}}_n = 0 \]

\[ \mathbf{\hat{a}}_y \times \mathbf{\hat{a}}_y = |\mathbf{\hat{a}}_y| |\mathbf{\hat{a}}_y| \sin(0) \mathbf{\hat{a}}_n = 0 \]

\[ \mathbf{\hat{a}}_z \times \mathbf{\hat{a}}_z = |\mathbf{\hat{a}}_z| |\mathbf{\hat{a}}_z| \sin(0) \mathbf{\hat{a}}_n = 0 \]

\[ \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_y = |\mathbf{\hat{a}}_x| |\mathbf{\hat{a}}_y| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = \mathbf{\hat{a}}_z \]

\[ \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_z = |\mathbf{\hat{a}}_x| |\mathbf{\hat{a}}_z| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = -\mathbf{\hat{a}}_y \]

\[ \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_x = |\mathbf{\hat{a}}_y| |\mathbf{\hat{a}}_x| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = -\mathbf{\hat{a}}_z \]

\[ \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_z = |\mathbf{\hat{a}}_y| |\mathbf{\hat{a}}_z| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = \mathbf{\hat{a}}_x \]

\[ \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_x = |\mathbf{\hat{a}}_z| |\mathbf{\hat{a}}_x| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = \mathbf{\hat{a}}_y \]

\[ \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_y = |\mathbf{\hat{a}}_z| |\mathbf{\hat{a}}_y| \sin(90^\circ) \mathbf{\hat{a}}_n = (1)(1)\mathbf{\hat{a}}_n = -\mathbf{\hat{a}}_x \]

Let \( \mathbf{B} = B_x \mathbf{\hat{a}}_x + B_y \mathbf{\hat{a}}_y + B_z \mathbf{\hat{a}}_z \) and \( \mathbf{A} = A_x \mathbf{\hat{a}}_x + A_y \mathbf{\hat{a}}_y + A_z \mathbf{\hat{a}}_z \), then

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\mathbf{\hat{a}}_x & \mathbf{\hat{a}}_y & \mathbf{\hat{a}}_z \\
A_x & A_y & A_z \\
B_x & B_y & B_z 
\end{vmatrix}
\]

\[
= (A_y B_z - A_z B_y) \mathbf{\hat{a}}_x - (A_x B_z - A_z B_x) \mathbf{\hat{a}}_y + (A_x B_y - A_y B_x) \mathbf{\hat{a}}_z
\]

\[
\left( \mathbf{a}_x \right) \times \left( \mathbf{a}_y \right) \times \left( \mathbf{a}_z \right) \neq \left( \mathbf{a}_x \right) \times \left( \mathbf{a}_z \right) \times \left( \mathbf{a}_y \right)
\]
CIRCULAR CYLINDRICAL COORDINATES

\( \rho, \varphi, z \) are coordinate variables which are mutually perpendicular.

Remember polar coordinates (The 2D version)

\[
\begin{align*}
  x &= \rho \cos(\varphi) \\
  y &= \rho \sin(\varphi)
\end{align*}
\]

\( \varphi \) is measured from x-axis toward y-axis.

Including the z-coordinate, we obtain the cylindrical coordinates (3D version)

A point is located by its \( \rho, \varphi \) and \( z \) coordinates. Or as the intersection of three mutually orthogonal surfaces.
1) Infinitely long cylinder of radius \( \rho = \rho_1 \).
2) Semi-infinite plane of constant angle \( \varphi = \varphi_1 \).
3) Infinite plane of constant elevation \( z = z_1 \).
The three unit vectors \( \hat{a}_z \), \( \hat{a}_\phi \), and \( \hat{a}_\rho \) are \textit{in the direction of increasing variables} and are perpendicular \( \perp \) to the surface at which the coordinate variable is constant.
Note that in Cartesian coordinates, unit vectors are not functions of coordinate variables. But in cylindrical coordinates $\hat{a}_\rho$, and $\hat{a}_\varphi$ are functions of $\varphi$.

The cylindrical coordinate system is *Right Handed*: $\hat{a}_\rho \times \hat{a}_\varphi = \hat{a}_z$. 
Increasing each coordinate variable by a differential amount $d\rho$, $d\phi$, and $dz$, one obtains:

Note that $\rho$ and $z$ are lengths, but $\phi$ is an angle which requires a metric coefficient to convert it to length.

\[ \text{arc length} = \rho \frac{d\phi}{\text{metric coefficient}} \]
Differential volume: \( dv = \rho d\rho d\varphi dz \)

Differential Surfaces: Six planes with differential areas shown in the figure above. (Try it!)
Transformations between Cylindrical and Cartesian Coordinates

From cylindrical to cart:
\[ x = \rho \cos(\varphi) \]
\[ y = \rho \sin(\varphi) \]
\[ z = z \]

From cart. To cyl.:
\[ \rho = \sqrt{x^2 + y^2} \]
\[ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \]
\[ z = z \]
Consider a vector in rectangular coordinates:

\[ \mathbf{E} = E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z \]

Wishing to write \( \mathbf{E} \) in cylindrical coordinates:

\[ \mathbf{E} = E_\rho \mathbf{\hat{a}}_\rho + E_\phi \mathbf{\hat{a}}_\phi + E_z \mathbf{\hat{a}}_z \]

From the dot product:

\[ E_\rho = \mathbf{E} \cdot \mathbf{\hat{a}}_\rho \]

\[ E_\phi = \mathbf{E} \cdot \mathbf{\hat{a}}_\phi \]

\[ E_z = \mathbf{E} \cdot \mathbf{\hat{a}}_z \]

\[ E_\rho = \left( E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z \right) \cdot \mathbf{\hat{a}}_\rho \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\rho + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\rho + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\rho \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\rho + \frac{E_y}{\rho} \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\rho + \frac{E_z}{\rho} \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\rho \]

\[ E_\phi = \left( E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z \right) \cdot \mathbf{\hat{a}}_\phi \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\phi + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\phi + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\phi \]

\[ = \frac{E_x}{\rho} \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\phi + \frac{E_y}{\rho} \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\phi + \frac{E_z}{\rho} \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\phi \]

\[ E_z = \left( E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z \right) \cdot \mathbf{\hat{a}}_z \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_z + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_z + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_z \]

\[ = \frac{E_x}{\rho} \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_z + \frac{E_y}{\rho} \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_z + \frac{E_z}{\rho} \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_z \]
Clearly:

\[ \hat{a}_x \cdot \hat{a}_\rho = |\hat{a}_x||\hat{a}_\rho| \cos(\varphi) = \cos(\varphi) \]

\[ \hat{a}_y \cdot \hat{a}_\rho = |\hat{a}_y||\hat{a}_\rho| \cos(90^\circ - \varphi) = \sin(\varphi) \]

\[ \hat{a}_x \cdot \hat{a}_\varphi = |\hat{a}_x||\hat{a}_\varphi| \cos(90^\circ + \varphi) = \sin(\varphi) \]

\[ \hat{a}_y \cdot \hat{a}_\varphi = |\hat{a}_y||\hat{a}_\varphi| \cos(\varphi) = \cos(\varphi) \]

\[ \hat{a}_z \cdot \hat{a}_\rho = |\hat{a}_z||\hat{a}_\rho| \cos(90^\circ) = 0 \]

\[ \hat{a}_z \cdot \hat{a}_\varphi = |\hat{a}_z||\hat{a}_\varphi| \cos(90^\circ) = 0 \]

So:

\[ E_\rho = E_x \cos(\varphi) + E_y \sin(\varphi) \]

\[ E_\varphi = -E_x \sin(\varphi) + E_y \cos(\varphi) \]

Or in matrix form

\[
\begin{bmatrix}
E_\rho \\
E_\varphi
\end{bmatrix}
= 
\begin{bmatrix}
\cos(\varphi) & \sin(\varphi) \\
-\sin(\varphi) & \cos(\varphi)
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\]
And, the inverse relation is:

\[
\begin{bmatrix}
  E_x \\
  E_y
\end{bmatrix} = \begin{bmatrix}
  \cos(\phi) & -\sin(\phi) \\
  \sin(\phi) & \cos(\phi)
\end{bmatrix}
\begin{bmatrix}
  E_\rho \\
  E_\phi
\end{bmatrix}
\]

Note that the story is not finished here, after transforming the components; you should also transform the coordinate variables.

<table>
<thead>
<tr>
<th>(a_\rho)</th>
<th>(a_\phi)</th>
<th>(a_z)</th>
</tr>
</thead>
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<tr>
<td>(a_x)</td>
<td>(\cos \phi)</td>
<td>(-\sin \phi)</td>
</tr>
<tr>
<td>(a_y)</td>
<td>(\sin \phi)</td>
<td>(\cos \phi)</td>
</tr>
<tr>
<td>(a_z)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 1.3

Transform the vector \(B = y a_x - x a_y + z a_z\) into cylindrical coordinates.

**Solution.** The new components are

\[
B_\rho = B \cdot a_\rho = y(a_x \cdot a_\rho) - x(a_y \cdot a_\rho) = y \cos \phi - x \sin \phi = \rho \sin \phi \cos \phi - \rho \cos \phi \sin \phi = 0
\]
\[
B_\phi = B \cdot a_\phi = y(a_x \cdot a_\phi) - x(a_y \cdot a_\phi) = -y \sin \phi - x \cos \phi = -\rho \sin^2 \phi - \rho \cos^2 \phi = -\rho
\]

Thus,

\[
B = -\rho a_\phi + z a_z
\]
THE SPHERICAL COORDINATE SYSTEM

$r, \theta, \varphi$ are coordinate variables.

$\varphi$ is measured from x-axis toward y-axis, and $\theta$ is measured from the z-axis toward the xy plane.

A general point is located by its coordinate variables $r, \theta, \varphi$, or as the intersection of three mutually perpendicular surfaces.
1) Sphere of radius $r = r_1$, centered at the origin.
2) Semi-infinite plane of constant angle $\varphi = \varphi_1$ with its axis aligned with z-axis.
3) Right angular cone with its apex centered at the origin, and its axis aligned with z-axis, and a cone angle $\theta = \theta_1$.

The three unit vectors $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\varphi$ are in the direction of increasing variables and are perpendicular $\perp$ to the surface at which the coordinate variable is constant.
Note that in spherical coordinates, unit vectors are functions of coordinate variables. \( \hat{a}_\rho, \hat{a}_\theta \) and \( \hat{a}_\phi \) are functions of \( \phi \) and \( \theta \).
The spherical coordinate system is *Right Handed*:

\[ \hat{a}_r \times \hat{a}_\theta = \hat{a}_\varphi \]

Increasing each coordinate variable by a differential amount \( dr \), \( d\theta \), and \( d\varphi \), one obtains:
Note that \( r \) is length, but \( \theta \) and \( \varphi \) are angles which requires a metric coefficient to convert them to lengths.

\[
\text{arc length} = r \, d\theta
\]

\[
\text{arc length} = r \frac{\sin \theta}{2} \, d\varphi
\]

Differential volume: \( dv = r^2 \sin \theta \, dr \, d\theta \, d\varphi \)

Differential Surfaces: Six surfaces with differential areas shown in the figure. (Try itttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttttt
Consider a vector in rectangular coordinates:

\[ \mathbf{E} = E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z \]

Wishing to write \( \mathbf{E} \) in spherical coordinates:

\[ \mathbf{E} = E_r \mathbf{\hat{a}}_r + E_\theta \mathbf{\hat{a}}_\theta + E_\phi \mathbf{\hat{a}}_\phi \]

From the dot product:

\[ E_r = \mathbf{E} \cdot \mathbf{\hat{a}}_r \quad E_\theta = \mathbf{E} \cdot \mathbf{\hat{a}}_\theta \quad E_\phi = \mathbf{E} \cdot \mathbf{\hat{a}}_\phi \]

\[ E_r = (E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z) \cdot \mathbf{\hat{a}}_r \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_r + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_r + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_r \]

\[ E_\phi = (E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z) \cdot \mathbf{\hat{a}}_\phi \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\phi + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\phi + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\phi \]

\[ E_\theta = (E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y + E_z \mathbf{\hat{a}}_z) \cdot \mathbf{\hat{a}}_\theta \]

\[ = E_x \mathbf{\hat{a}}_x \cdot \mathbf{\hat{a}}_\theta + E_y \mathbf{\hat{a}}_y \cdot \mathbf{\hat{a}}_\theta + E_z \mathbf{\hat{a}}_z \cdot \mathbf{\hat{a}}_\theta \]
From figure

\[ \hat{a}_z \cdot \hat{a}_r = |\hat{a}_z| |\hat{a}_r| \cos(\theta) = \cos(\theta) \]

\[ \hat{a}_\rho \cdot \hat{a}_r = |\hat{a}_\rho| |\hat{a}_r| \cos(90^\circ - \theta) = \sin(\theta) \]

\[ \hat{a}_z \cdot \hat{a}_\theta = |\hat{a}_z| |\hat{a}_\theta| \cos(90^\circ + \theta) = -\sin(\theta) \]

And the rest is left to you as an exercise!

So:

\[ E_r = E_x \sin(\theta) \cos(\phi), \quad E_y \sin(\theta) \sin(\phi) + E_z \cos(\theta) \]

Note that, after transforming the components; you should also transform the coordinate variables.

<table>
<thead>
<tr>
<th>( a_r )</th>
<th>( a_\theta )</th>
<th>( a_\phi )</th>
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<tbody>
<tr>
<td>( a_x \cdot )</td>
<td>( \sin \theta \cos \phi )</td>
<td>( \cos \theta \cos \phi )</td>
</tr>
<tr>
<td>( a_y \cdot )</td>
<td>( \sin \theta \sin \phi )</td>
<td>( \cos \theta \sin \phi )</td>
</tr>
<tr>
<td>( a_z \cdot )</td>
<td>( \cos \theta )</td>
<td>( -\sin \theta )</td>
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Example 1.4

We illustrate this transformation procedure by transforming the vector field \( \mathbf{G} = \left( \frac{xz}{y} \right) \mathbf{a}_x \) into spherical components and variables.

**Solution.** We find the three spherical components by dotting \( \mathbf{G} \) with the appropriate unit vectors, and we change variables during the procedure:

\[
G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y} \sin \theta \cos \phi \\
= r \sin \theta \cos \theta \frac{\cos^2 \phi}{\sin \phi} \\

G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y} \cos \theta \cos \phi \\
= r \cos^2 \theta \frac{\cos^2 \phi}{\sin \phi} \\

G_\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y} \left( -\sin \phi \right) \\
= -r \cos \theta \cos \phi
\]

Collecting these results, we have

\[
\mathbf{G} = r \cos \theta \cos \phi \left( \sin \theta \cot \phi \mathbf{a}_r + \cos \theta \cot \phi \mathbf{a}_\theta - \mathbf{a}_\phi \right)
\]