ELECTRIC FLUX DENSITY

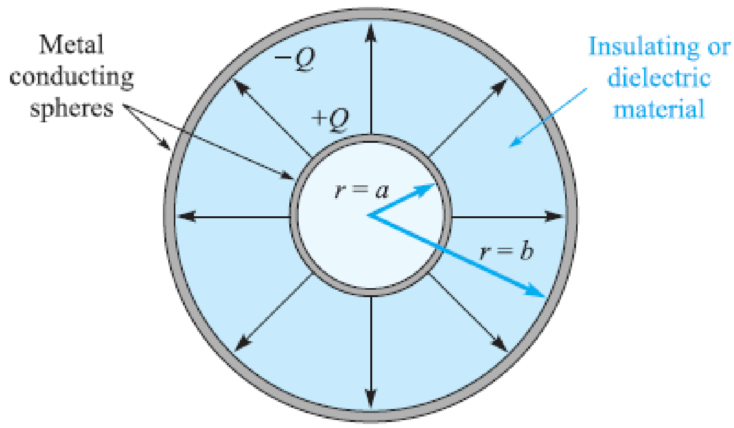

Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of D are not functions of the dielectric between the spheres.

$$\boxed{\Psi = Q}$$

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r}$$

$$\boxed{\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r}$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \qquad \text{(free space only)}$$

$$\boxed{\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi \epsilon_0 R^2}\mathbf{a}_R} \qquad \text{(free space only)}$$

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R}$$

GAUSS'S LAW

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

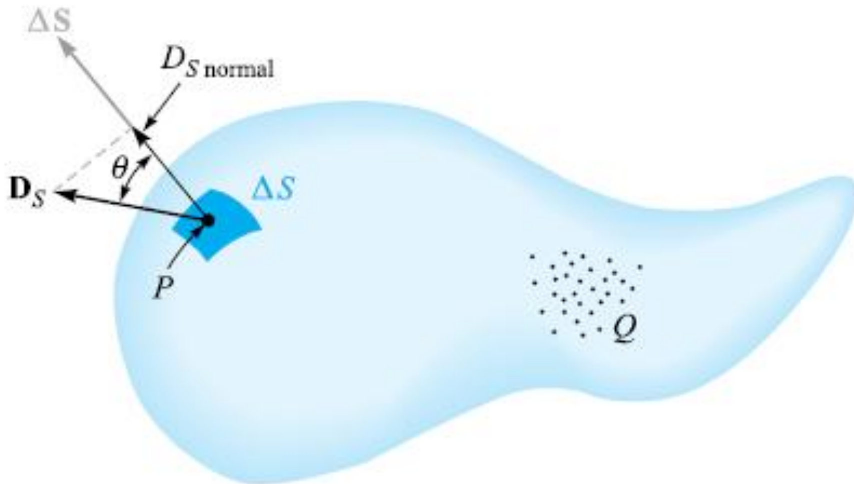

Figure 3.2 The electric flux density $\mathbf{D}_S$ at P arising from charge Q. The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta\mathbf{S}$.

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta\, \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

$$\boxed{\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q}$$

$$Q = \Sigma Q n$$

$$Q = \int \rho_L \, dL$$

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

$$\boxed{\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv}$$

EXAMPLE 3.1

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a.

Solution. We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta\, d\theta\, d\phi = a^2 \sin\theta\, d\theta\, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta\, d\theta\, d\phi\ \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta\, d\theta\, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi$$

$$\int_0^{2\pi} \frac{Q}{4\pi}\left(-\cos\theta\right)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

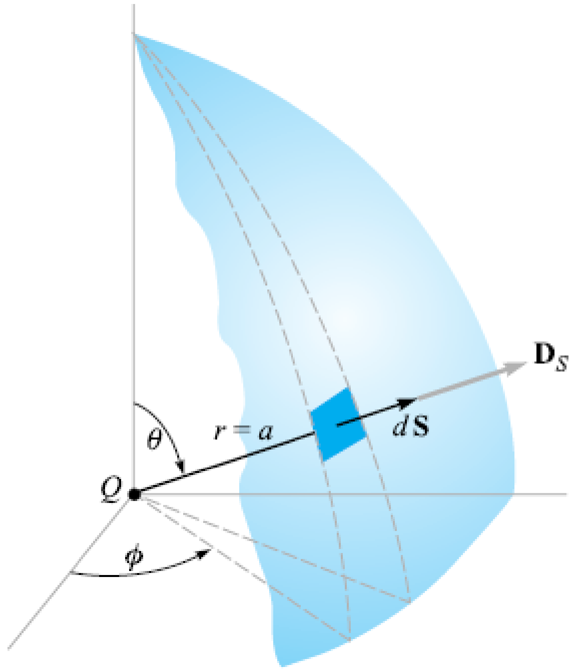

Figure 3.3 Applying Gauss's law to the field of a point charge Q on a spherical closed surface of radius a. The electric flux density $\mathbf{D}$ is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

$$\boxed{Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}}$$

Choose a closed surface which satisfies two conditions:

1. $\boldsymbol{D_s}$ is everywhere either normal or tangential to the closed surface, so that $\boldsymbol{D_s}\cdot\boldsymbol{ds}$ becomes either $D_s ds$ or zero, respectively.
2. On that portion of the closed surface for which $\boldsymbol{D_s}\cdot\boldsymbol{ds}$ is not zero, $D_s = constant$.

if we cannot show that symmetry exists then we cannot use Gauss's law

Example (Flux density of infinite uniform line charge):

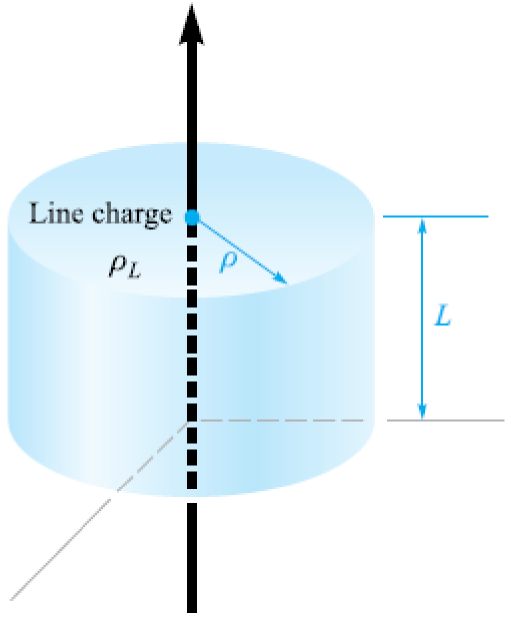

Figure 3.4 The gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ. $\mathbf{D}$ is constant in magnitude and everywhere perpendicular to the cylindrical surface; $\mathbf{D}$ is parallel to the end faces.

$$\boldsymbol{D} = D_\rho(\rho)\hat{\boldsymbol{a}}_\rho$$

A closed right circular cylinder of radius ρextending from $z = 0$ to $z = L$ is choosen as Gaussian surface and shown in Figure 3.4.

$$Q = \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L$$

$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

$$Q = \rho_L L$$

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Example (Flux density of Coaxial Cable):

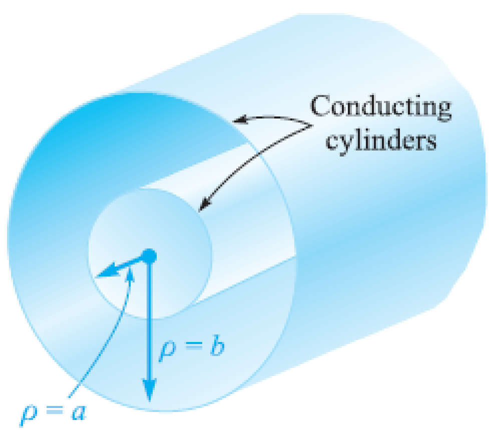

Figure 3.5 The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

Two coaxial cylindrical conductors.
The inner of radius a and the outer of radius b, each infinite in extent.
A charge distribution of ρ_s on the outer surface of the inner conductor.

$$\boldsymbol{D} = D_\rho(\rho)\hat{\boldsymbol{a}}_\rho$$

A right *circular cylinder* of length L and radius ρ, where $a < \rho < b$, is necessarily chosen as the *gaussian surface*;

$$Q = D_S 2\pi\rho L$$

The total charge on a length $L$of the inner conductor is

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \qquad (a < \rho < b)$$

This result might be expressed *in terms of charge per unit length* because the innerconductor has$2\pi a\rho_s$coulombs on a meter length, and hence, letting$\rho_L = 2\pi a\rho_s$,

$$\boxed{\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho}$$

Identical infinite line charge.

Thetotal charge on outer surface must be:

$$Q_{outer\ cyl} = -2\pi a L\rho_{S,inner\ cyl}$$

The surface charge on the outer cylinder is found as:

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

Or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

Choosing a cylinder of radius $\rho, \rho > b$, for thegaussian surface:

$$0 = D_S 2\pi\rho L \qquad (\rho > b)$$

$$D_S = 0 \qquad (\rho > b)$$

An identical result would be obtained for $\rho < a$.

- Thus the coaxial cable orcapacitor has no external field (we have proved that the outer conductor is a *"shield"*),and there is no field within the center conductor.

- Our result is also useful for a *finite* length of coaxial cable, open at both ends, providedthe length *L* is many times greater than the radius *b* so that the nonsymmetricalconditions at the two ends do notappreciably affect the solution. Such a device isalso termed a *coaxial capacitor*.

EXAMPLE 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55\ \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outer cyl}} = \frac{Q_{\text{outer cyl}}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39\ \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho}\ \text{nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{1079}{\rho}\ \text{V/m}$$

Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

APPLICATION OF GAUSS'S LAW:
DIFFERENTIAL VOLUME ELEMENT (Introducing the Divergence)

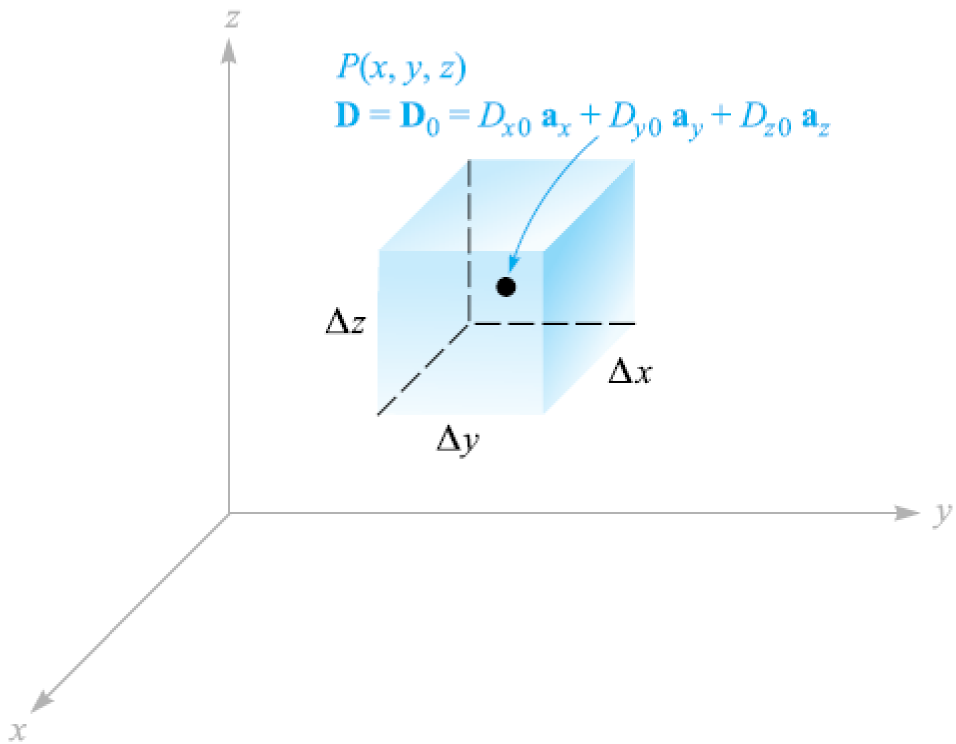

Figure 3.6 A differential-sized gaussian surface about the point *P* is used to investigate the space rate of change of **D** in the neighborhood of *P*.

Choose a closed surface the ***small*** rectangular box, centered at *P*, having sides of lengths Δx, Δy, and Δz.

Assume**D** is ***almost constant*** over the surface.

To obtaininformation about the way **D** varies in the region of our small surface.

The value of **D** at the point *P*

$$\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$$

Apply Gauss's law

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

$$\begin{aligned}\int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x \\ &\doteq D_{x,\text{front}} \Delta y\, \Delta z\end{aligned}$$

Since the surface element is very small, **D** is essentially constant (over*this* portion of the entire closed surface).

Approximate the value of $D_x$at this front face. The front face is at a distance of$\frac{\Delta x}{2}$ from *P(the small change in* ***D*** *may be adequately represented by using the first two terms of the Taylor's-series expansion for* ***D****.)*

$$\begin{aligned}D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\end{aligned}$$

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y\, \Delta z$$

For the back surface

$$\begin{aligned}\int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta\mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y\, \Delta z\, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y\, \Delta z\end{aligned}$$

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y\, \Delta z$$

Adding;

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x\, \Delta y\, \Delta z$$

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x\, \Delta y\, \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x\, \Delta y\, \Delta z$$

Collecting all results;

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x\, \Delta y\, \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$\boxed{\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v}$$

EXAMPLE 3.3

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y\, \mathbf{a}_x - e^{-x} \cos y\, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is 10^{-9} m^3, then we have enclosed about 2 nC.

DIVERGENCE
AND MAXWELL'S FIRST EQUATION

From previous discussion, allowing the volume element Δv to shrink to zero

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}=\rho_v$$

This could be done for any vector **A** to find $\oint_s \boldsymbol{A}.\boldsymbol{ds}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

The divergence of **A** is defined mathematically as;

$$\text{Divergence of }\mathbf{A}=\text{div }\mathbf{A}=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

Definition: ***The divergence of the vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.***

A positive divergence for any vector quantity indicates a ***source*** of that vector quantity at that point. Similarly, a negative divergence indicates a ***sink***.

$$\text{div }\mathbf{D}=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)\qquad\text{(rectangular)}$$

$$\text{div }\mathbf{D}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho)+\frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi}+\frac{\partial D_z}{\partial z}\qquad\text{(cylindrical)}$$

$$\text{div }\mathbf{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}\qquad\text{(spherical)}$$

The divergence is an operation which is performed on a vector, but that the result is a scalar.

EXAMPLE 3.4

Find div **D** at the origin if $\mathbf{D} = e^{-x}\sin y\,\mathbf{a}_x - e^{-x}\cos y\,\mathbf{a}_y + 2z\mathbf{a}_z$.

Solution. We use (10) to obtain

$$\begin{aligned}\text{div}\,\mathbf{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\ &= -e^{-x}\sin\,y + e^{-x}\sin\,y + 2 = 2\end{aligned}$$

The value is the constant 2, regardless of location.
If the units of **D** are C/m^2, then the units of div **D** are C/m^3. This is a volume charge density, a concept discussed in the next section.

D3.7. In each of the following parts, find a numerical value for div **D** at the point specified: (*a*) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z$ C/m^2 at $P_A(2, 3, -1)$; (*b*) $\mathbf{D} = 2\rho z^2\sin^2\phi\,\mathbf{a}_\rho + \rho z^2\sin 2\phi\,\mathbf{a}_\phi + 2\rho^2 z\sin^2\phi\,\mathbf{a}_z$ C/m^2 at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$; (*c*) $\mathbf{D} = 2r\sin\theta\cos\phi\,\mathbf{a}_r + r\cos\theta\cos\phi\,\mathbf{a}_\theta - r\sin\phi\,\mathbf{a}_\phi$ C/m^2 at $P_C(r = 1.5, \theta = 30^\circ, \phi = 50^\circ)$.

Ans. −10.00; 9.06; 1.29

Combine the following Eqs.;

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)$$

To obtain

$$\boxed{\text{div}\,\mathbf{D} = \rho_v}$$

This is the first of Maxwell's four equations as they apply to electrostatics andsteady magnetic fields, and it states that the *electric flux per unit volume leaving avanishingly smallvolume unit is exactly equal to the volume charge density there.* This equation is called the *point form of Gauss's law*. *(per-unit-volume basis for a vanishingly small volume, orat a point.)*

Illustration, the divergence of **D**in the region abouta point charge*Q* located at the origin.

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{Q}{4\pi r^2}\right) = 0 \qquad (\text{if } r \neq 0)$$

Thus, $\rho_v = 0$ everywhere except at the origin, where it is infinite.

The divergence operation is not limited to electric flux density; it can be appliedto any vector field.

THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$\boxed{\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div}(\mathbf{D})}$$

From previous analysis, we know that:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

The first and last expressions constitute the divergence theorem,which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal tothe integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

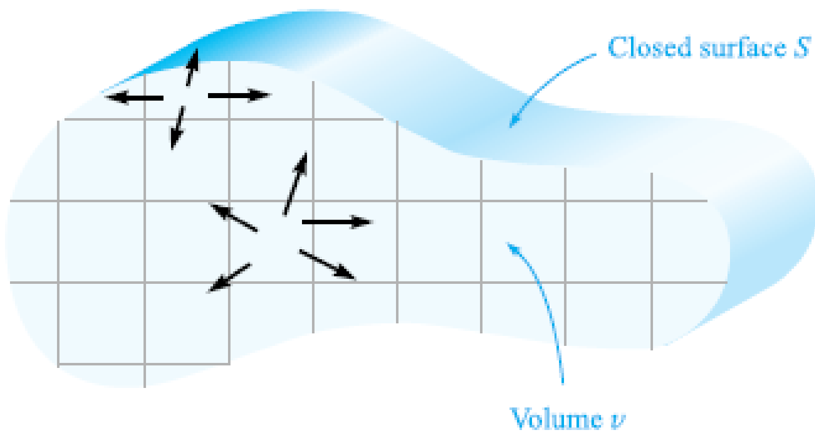

Figure 3.7 The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

EXAMPLE 3.5

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that $\mathbf{D}$ is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$
$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3 \int_0^2 (D_x)_{x=0} dy\, dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz$$
$$- \int_0^3 \int_0^1 (D_y)_{y=0} dx\, dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx\, dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$
$$= \int_0^3 4\, dz = 12$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y\, dx\, dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$
$$= \int_0^3 4\, dz = 12$$