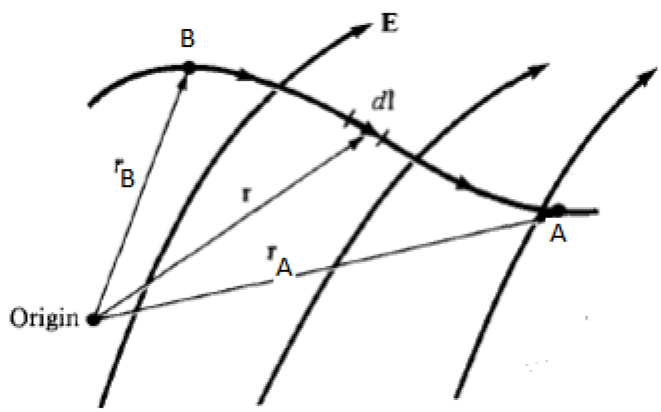
Energy and Potential

ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD



Suppose we wish to move a charge *Q* a distance $d\boldsymbol{L}$ in an electric field $\boldsymbol{E}$. The force on *Q* arising from the electric field is

$$\mathbf{F}_E = Q\mathbf{E}$$

The component of this force in the direction *d***L** which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

The differential work done by an external source moving charge *Q* is

$$dW = -Q\mathbf{E} \cdot d\mathbf{L}$$

This differential amount of work required may be zero under several conditions:

- There are the trivial conditions for which $\boldsymbol{E}$, Q, or $d\boldsymbol{L}$ is zero.
- $\boldsymbol{E}$ and $d\boldsymbol{L}$ are perpendicular, charge is moved always in a direction at right angles to the electric field.

Energy must be expended to move against the field.

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The work required to move the charge a finite distance must be determined by integrating,

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

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THE LINE INTEGRAL

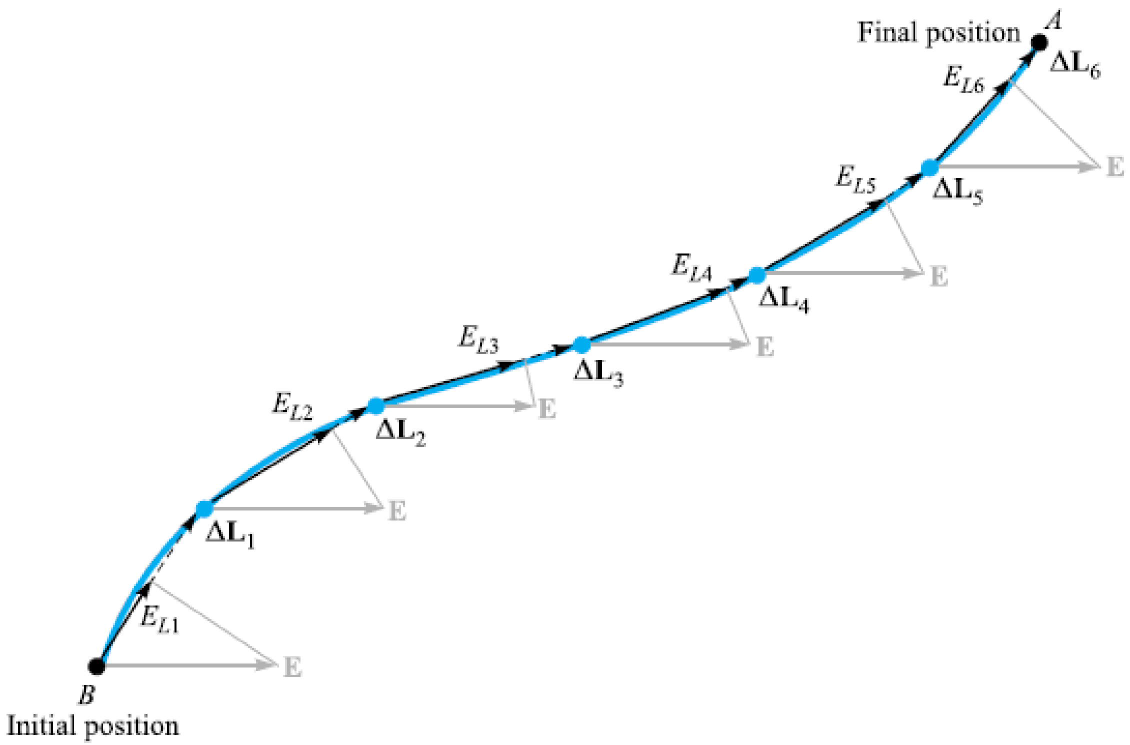

Figure 4.1 A graphical interpretation of a line integral in a uniform field. The line integral of **E** between points *B* and *A* is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

Without using vector analysis we should have to write

$$W = -Q \int_{\text{init}}^{\text{final}} E_L \, dL$$

whereE_L = component of **E** along $d\mathbf{L}$.

Assume a *uniform electric field*is selected for simplicity. The path is divided into six segments $\Delta \boldsymbol{L}1, \Delta \boldsymbol{L}2, \ldots, \Delta \boldsymbol{L}6$, and the components of **E** along each segment are denoted by $EL1, EL2, \ldots, EL6$. The work involved in moving a charge *Q* from *B* to *A* is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$
$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$
$$W = -Q\mathbf{E} \cdot (\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)$$

- Work involved in moving the charge depends only on *Q*, **E**, and $\boldsymbol{L_{BA}}$, a vector drawnfrom the initial to the final point of the path chosen.
- It does not dependon the particularpath we have selected along which to carry the charge.

Remember:

$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$	(rectangular)
$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z$	(cylindrical)
$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$	(spherical)

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EXAMPLE 4.1

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2C from $B(1, 0, 1)$ to $A(0.8, 0.6, 1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \quad z = 1$$

Solution. We use $W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$, where $\mathbf{E}$ is not necessarily constant. Working in rectangular coordinates, the differential path $d\mathbf{L}$ is $dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$, and the integral becomes

$$\begin{aligned} W &= -Q\int_B^A \mathbf{E}\cdot d\mathbf{L} \\ &= -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z)\cdot(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \\ &= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz \end{aligned}$$

where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variable of integration. Using the equation of the circular path (and selecting the sign of the radical which is correct for the quadrant involved), we have

$$\begin{aligned} W &= -2\int_1^{0.8} \sqrt{1-x^2}\,dx - 2\int_0^{0.6} \sqrt{1-y^2}\,dy - 0 \\ &= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_0^{0.6} \\ &= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) \\ &= -0.96\ \text{J} \end{aligned}$$

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EXAMPLE 4.2

Again find the work required to carry 2C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. We start by determining the equations of the straight line. Any two of the following three equations for planes passing through the line are sufficient to define the line:

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first equation we have

$$y = -3(x - 1)$$

and from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^{1} dz$$

$$= 6\int_1^{0.8} (x-1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$

$$= -0.96 \text{ J}$$

Example☹ (infinite Line Charge) As a final example illustrating the evaluation of the line integral, weinvestigateseveral paths that we might take near an infinite linecharge. The field has beenobtained several times and is entirely in the radial direction;

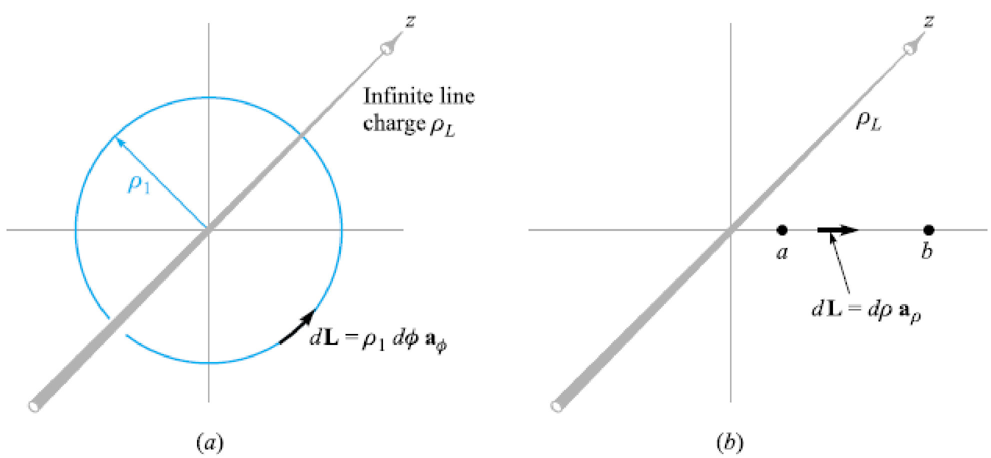

Figure 4.2 (a) A circular path and (b) a radial path along which a charge of Q is carried in the field of an infinite line charge. No work is expected in the former case.

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The work done in carrying the positive charge *Q* about a circular path of radius $\rho_b$centered at the line charge, as illustrated in Figure 4.2*a*.

The **work** must be **nil**, for the path is always perpendicular to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it.

$d\rho$anddzbe zero, so $d\boldsymbol{L} = \rho_1 d\varphi \hat{\boldsymbol{a}}_{\boldsymbol{\varphi}}$

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\, d\phi\, \mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\, \mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

We will now carry the charge from $\rho = a$to $\rho = b$ along a radial path (Figure 4.2*b*). Here $d\boldsymbol{L} = d\rho\, \hat{\boldsymbol{a}}\rho$and

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho = -Q\int_a^b \frac{\rho_L}{2\pi\epsilon_0}\frac{d\rho}{\rho}$$

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Because b is larger than a, $ln(b/a)$ is positive, and *the work done is negative*, indicating that the external source that is moving the charge receives energy.

DEFINITION OF POTENTIALDIFFERENCE AND POTENTIAL

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

potential difference $\boldsymbol{V}$: the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$\text{Potential difference } = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

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V_{AB}signifies the potential difference between points $A$and $B$and is the work done inmoving the unit charge from B (last named) to A (first named).

Thus, in determining V_{AB}, B is the initial point and $A$is the final point.

Potential difference is measured in joules per coulomb, for which the *volt* isdefined as a more common unit, abbreviated as V.

Hence the potential differencebetween points A and B is

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} \text{ V}$$

From the line-charge example of Section 4.2 we found that the work done intaking a charge Q from $\rho = b$to $\rho = a$was

$$W = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Thus, the potential difference between points at $\rho = a$ and $\rho = b$ is

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

*Potential difference in the field of a point charge:*We can try out this definition by finding the potential difference between points$A$ and B at radial distances $r_A$and $r_B$from a point charge Q. Choosing an origin at Q,

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

$$d\mathbf{L} = dr\, \mathbf{a}_r$$

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$V_{AB} = V_A - V_B$$

THE POTENTIAL FIELDOF A POINT CHARGE

The potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin:

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$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

Define $V = 0$ at infinity.

Let the point at $r = r_B$ recede to infinity, the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

Or

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

$Q/4\pi\epsilon_o r$ joules of work must be done in carrying a unit charge from infinity to any point *r* meters from the charge *Q*.

Expressing the potential without selecting a specific zero reference is accomplished by identifying r_A as r and letting $Q/4\pi\epsilon_o r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1$$

C_1 may be selected so that $V = 0$ at any desired value of *r*. We could also select the zero reference indirectly by electing to let V be V_o at $r = r_o$.

Equipotential surface is a surface composed of all those points having the same value of potential.

- All field lines would be perpendicular to an equipotential surface at the points where they intersect it.
- No work is involved in moving a unit charge around on an equipotential surface.
- The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

THE POTENTIAL FIELD OF A SYSTEM OF CHARGES: CONSERVATIVE PROPERTY

The potential field of a single point charge Q_1 and locate at $\boldsymbol{r_1}$, for a zero reference at infinity;

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$

The potential arising from n point charges is;

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$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|}$$

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_v \Delta v$, then;

$$V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \cdots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

As the number of elements to become infinite;

$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

- $V(\boldsymbol{r})$ is determined with respect to a zero reference potential at infinity.
- $|\boldsymbol{r} - \boldsymbol{r}'|$ is that distance from the source point to the field point.

For a line charge or a surface charge distribution:

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\,dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\,dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

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EXAMPLE 4.3

To illustrate the use of one of these potential integrals, we will find V on the z axis for a uniform line charge ρ_L in the form of a ring, $\rho = a$, in the $z = 0$ plane, as shown in Figure 4.3.

Solution. Working with Eq. (18), we have $dL' = a\,d\phi'$, $\mathbf{r} = z\mathbf{a}_z$, $\mathbf{r}' = a\mathbf{a}_\rho$, $|\mathbf{r}-\mathbf{r}'| = \sqrt{a^2+z^2}$, and

$$V = \int_0^{2\pi} \frac{\rho_L a\,d\phi'}{4\pi\epsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}}$$

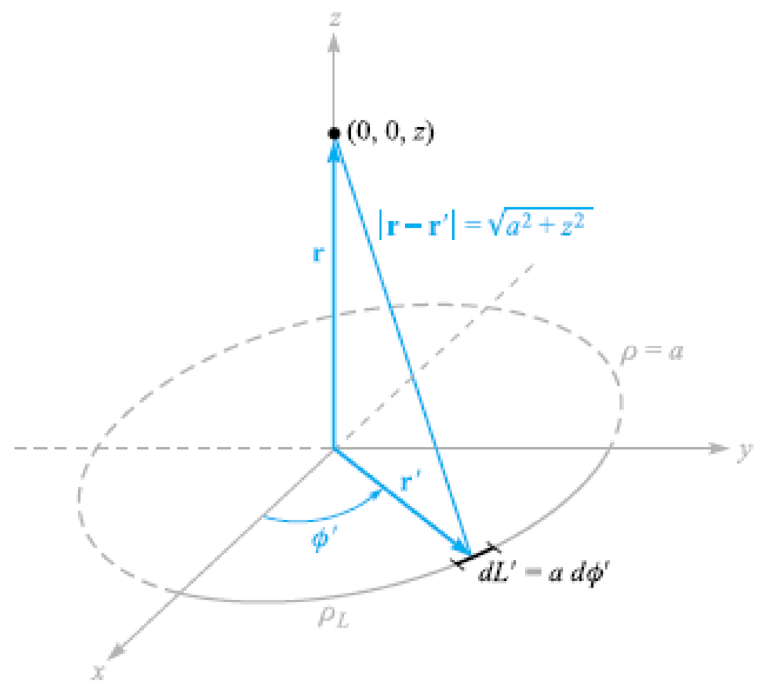

Figure 4.3 The potential field of a ring of uniform line charge density is easily obtained from $V = \int \rho_L(\mathbf{r}')\,dL'/(4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|)$.

The expression for potential (zero reference at infinity),

$$V_A = -\int_{\infty}^{A} \mathbf{E}\cdot d\mathbf{L}$$

or potential difference,

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E}\cdot d\mathbf{L}$$

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which is not dependent on the path chosen, regardless of the source of the **E** field.
No work is done in carrying the unit charge around any *closed path*,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Conservative Property of static E

This is true for *static* fields, but not for time-varying fields.

Illustration: consider the dc circuit shown in Figure 4.4. Two points, *A* and *B*, are marked, and conservative property states that no work is involved in carrying a unit charge from *A* through *R*2 and *R*3 to *B* and back to *A* through *R*1, or that the sum of the potential differences around any closed path is zero.

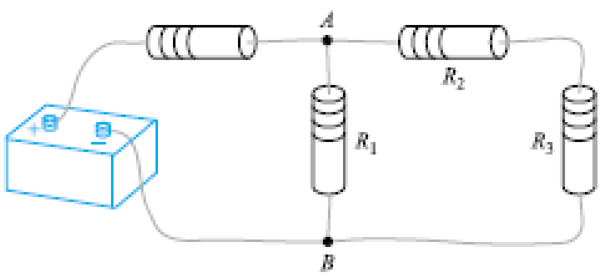

Figure 4.4 A simple dc-circuit problem that must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

More general form of Kirchhoff's circuital law for voltages.

Any field that satisfies an equation of the form of conservative property, (i.e., where the closed line integral of the field is zero) is said to be a ***conservative field***. (energy is *conserved in carrying* a charge around a closed path).

Illustration: consider the force field, $\boldsymbol{F} = \sin \pi\rho\, \hat{\boldsymbol{a}}\varphi$. Around a circular path of radius $\rho = \rho 1$, we have $d\boldsymbol{L} = \rho\, d\varphi\, \boldsymbol{a}\varphi$, and

$$\oint \mathbf{F} \cdot d\mathbf{L} = \int_0^{2\pi} \sin \pi\rho_1 \mathbf{a}_\phi \cdot \rho_1 d\phi\, \mathbf{a}_\phi = \int_0^{2\pi} \rho_1 \sin \pi\rho_1\, d\phi$$
$$= 2\pi\rho_1 \sin \pi\rho_1$$

The integral is zero if $\rho 1 = 1, 2, 3, \ldots$, etc., but it is not zero for other values of $\rho 1$, or for most other closed paths, and the given field is not conservative. A conservative field must yield a zero value for the line integral around every possible closed path.

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POTENTIAL GRADIENT

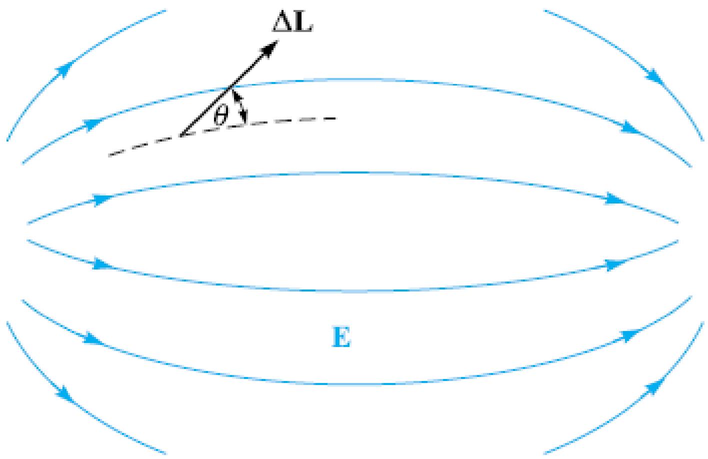

Figure 4.5 A vector incremental element of length ΔL is shown making an angle of θ with an E field, indicated by its streamlines. The sources of the field are not shown.

$$V = -\int \mathbf{E} \cdot d\mathbf{L}$$

For a short element of length $\Delta \boldsymbol{L}$ along which $\boldsymbol{E}$ is almost constant, leading to ΔV**:**

$$\Delta V \doteq -\mathbf{E} \cdot \Delta \mathbf{L}$$

$$\Delta V \doteq -E \Delta L \cos\theta$$

In a limiting since, the derivative dV/dL

$$\frac{dV}{dL} = -E\cos\theta$$

Question) In which direction should $\Delta \boldsymbol{L}$ be placed to obtain a maximum value of ΔV?

Answer) The maximum positive increment of potential, ΔV_{max}, will occur when $cos\ \theta = -1$, or $\Delta \boldsymbol{L}$ points in the direction *opposite* to **E**. For this condition,

$$\left.\frac{dV}{dL}\right|_{\max} = E$$

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1. The magnitude of the electric field intensity is given by the ***maximum*** value of the ***space rate of change of potential (with distance).***
2. This maximum value is obtained when the direction of the distance increment is opposite to **E** or, in other words, the *direction of **E** is opposite to the direction in which the potential is increasing the most rapidly*.

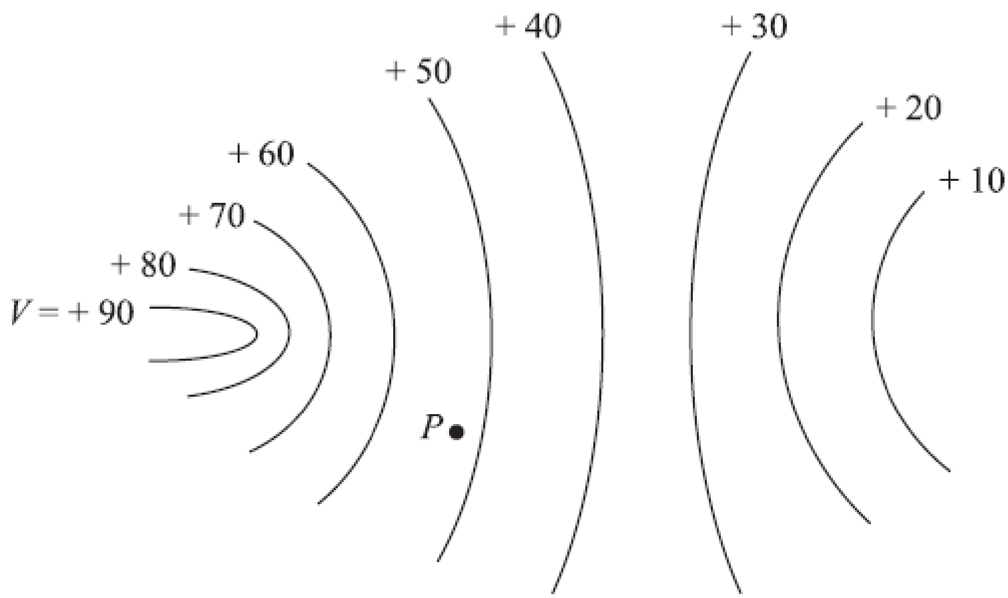

Figure 4.6 A potential field is shown by its equipotential surfaces. At any point the **E** field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.

The direction in which the potential is changing (increasing) the most rapidly is (From the sketch) to be left and slightly upward.

It seems likely that the direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of *increasing* potential), and this is correct, for if $\Delta \boldsymbol{L}$ is directed along an equipotential, $\Delta V = 0$ by our definition of an equipotential surface. But then**:**

$$\Delta V = -\mathbf{E} \cdot \Delta \mathbf{L} = 0$$

And as neither **E** nor $\Delta \boldsymbol{L}$ is zero, **E** must be perpendicular to this $\Delta \boldsymbol{L}$ or perpendicular to the equipotentials.

Letting $\hat{\boldsymbol{a}}_N$ be a unit vector normal to the equipotential surface and directed toward the higher potentials.

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The electric field intensity is then expressed in terms of the potential,

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N$$

which shows that:

*the magnitude of **E** is given by the maximum space rate of change of V and the direction of **E** is normal to the quipotential surface (in the direction of decreasing potential).*

Because $dV/dL|max$ occurs when $\Delta \boldsymbol{L}$ is in the direction of $\hat{\boldsymbol{a}}_N$, this fact is expressed by writing

$$\left.\frac{dV}{dL}\right|_{\max} = \frac{dV}{dN}$$

And

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N$$

The operation on V by which $-\boldsymbol{E}$ is obtained is known as the ***gradient***, and the gradient of a scalar field T is defined as:

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\mathbf{a}_N$$

$$\mathbf{E} = -\text{grad } V$$

Taking the total differential of V

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

But we also have

$$dV = -\mathbf{E}\cdot d\mathbf{L} = -E_x\,dx - E_y\,dy - E_z\,dz$$

Because both expressions are true for any dx, dy, and dz, then

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$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially to yield

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

To evaluate the gradient in rectangular coordinates

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

The del vector operator may be used

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

This allows us to use a very compact expression to relate $\boldsymbol{E}$ and V

$$\mathbf{E} = -\nabla V$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(rectangular)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

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EXAMPLE 4.4

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \text{ V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \text{ V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$$
$$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\,\mathbf{a}_x - 17.71x^2\,\mathbf{a}_y + 44.3\,\mathbf{a}_z \text{ pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla \cdot \mathbf{D} = -35.4y \text{ pC/m}^3$$

At P, $\rho_v = -106.2 \text{ pC/m}^3$.

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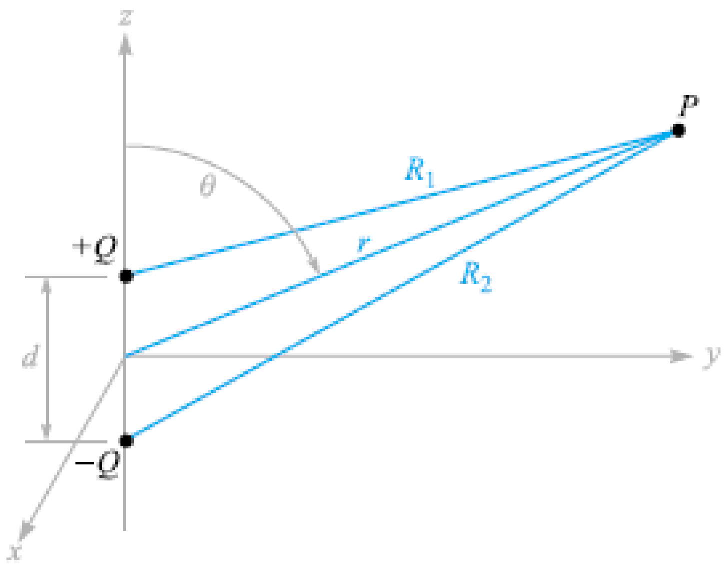
THE ELECTRIC DIPOLE

An *electric dipole*, or simply a *dipole*, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to the distance to the point *P* at which we want to know the electric and potential fields.



The distant point *P* is described by the spherical coordinates *r, θ,* and $\varphi = 90^\circ$, in view of the azimuthal symmetry.

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\,\frac{R_2 - R_1}{R_1 R_2}$$

$z = 0$ plane is at zero potential, as are all points at infinity.

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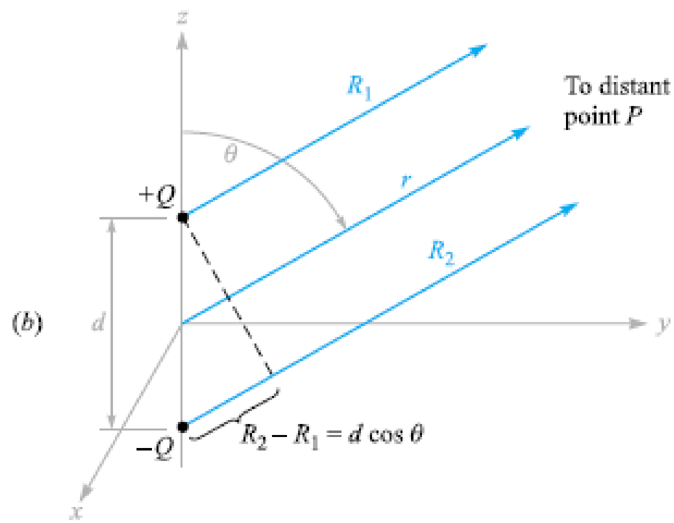

Figure 4.8 (a) The geometry of the problem of an electric dipole. The dipole moment $p = Qd$ is in the $\mathbf{a}_z$ direction. (b) For a distant point P, R_1 is essentially parallel to R_2, and we find that $R_2 - R_1 = d\cos\theta$.

For numerator, we use the approximation

$$R_2 - R_1 \doteq d\cos\theta$$

And for the denominator, we may use

$$R_1 \approx R_2 \approx r$$

Then;

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

$$\mathbf{E} = -\left(-\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\mathbf{a}_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\mathbf{a}_\theta\right)$$

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

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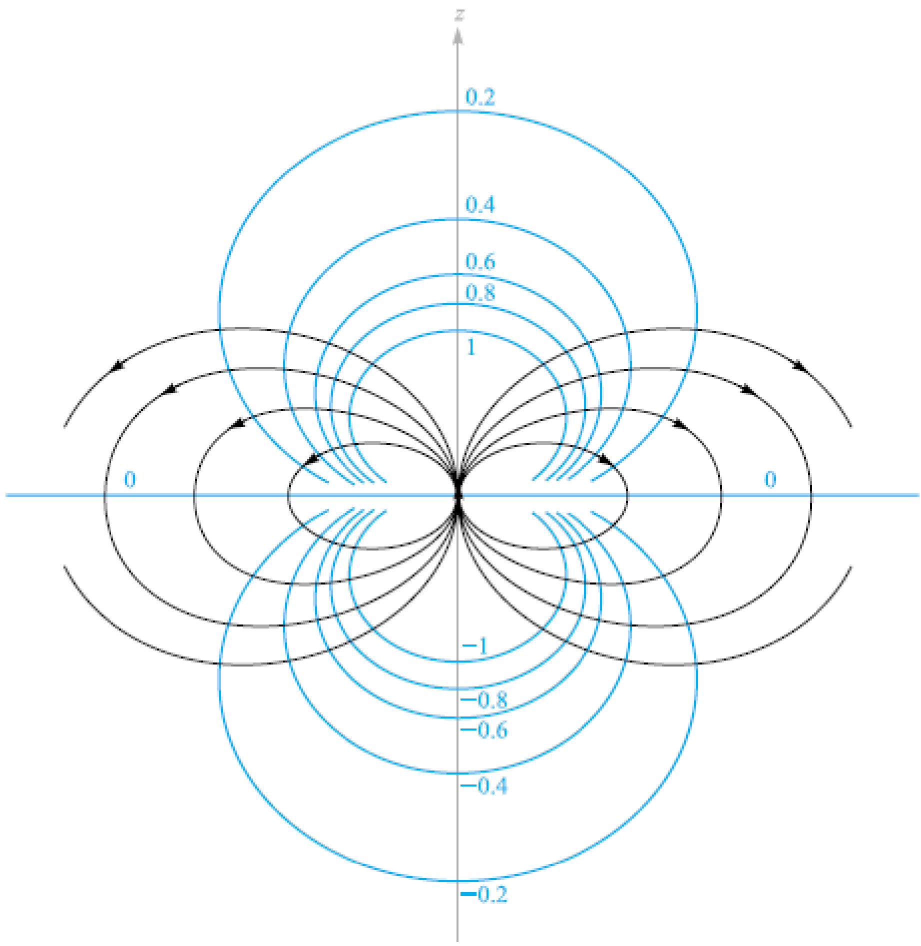

Figure 4.9 The electrostatic field of a point dipole with its moment in the $\mathbf{a}_z$ direction. Six equipotential surfaces are labeled with relative values of V.

If the ***vector length*** directed from $-Q$ to $+Q$ *is defined* as ***d***.
Define the *dipole moment* as $Q\mathbf{d}$ and assign it the symbol $\boldsymbol{p}$, then

$$\mathbf{p} = Q\mathbf{d}$$

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\epsilon_0 r^2}$$

$$V = \frac{1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2}\mathbf{p} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

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ENERGY DENSITY IN THE ELECTROSTATIC FIELD

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots) = \frac{1}{2}\sum_{m=1}^{m=N} Q_m V_m$$

V_m is the potential at the location of Q_m due to all point charges except Q_m itself.

For continuous charge distribution, each charge is replaced by $\rho_v dv$, and the summation becomes an integral

$$W_E = \frac{1}{2}\int_{\text{vol}} \rho_v V\, dv$$

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\, dv = \frac{1}{2}\int_{\text{vol}} \epsilon_0 E^2\, dv$$

Illustration: Calculate the energy stored in the electrostatic field of a section of a coaxial cable or capacitor of length L.

We found previously that

$$D_\rho = \frac{a\rho_S}{\rho}$$

$$\mathbf{E} = \frac{a\rho_S}{\epsilon_0\rho}\mathbf{a}_\rho$$

ρ_s is the surface charge density on the inner conductor, whose radius is a.

$$W_E = \frac{1}{2}\int_0^L\int_0^{2\pi}\int_a^b \epsilon_0\frac{a^2\rho_S^2}{\epsilon_0^2\rho^2}\rho\, d\rho\, d\phi\, dz = \frac{\pi L a^2\rho_S^2}{\epsilon_0}\ln\frac{b}{a}$$

Another way of solving the problem: choose the outer conductor as zero-potential reference, and the potential of the inner cylinder is then:

$$V_a = -\int_b^a E_\rho\, d\rho = -\int_b^a \frac{a\rho_S}{\epsilon_0\rho}\, d\rho = \frac{a\rho_S}{\epsilon_0}\ln\frac{b}{a}$$

Assuming the surface charge density ρ_s at $\rho = a$ to be a volume charge density $\rho_v = \rho_s/t$, extending from $\rho = a - t/2$ to $\rho = a + t/2$, then

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$$W_E = \frac{1}{2}\int_{\text{vol}} \rho_v\, V\, dV = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_{a-t/2}^{a+t/2} \frac{\rho_S}{t} a \frac{\rho_S}{\epsilon_0} \ln\frac{b}{a} \rho\, d\rho\, d\phi\, dz$$

So, again we obtain

$$W_E = \frac{a^2 \rho_S^2 \ln(b/a)}{\epsilon_0} \pi L$$

You can recognize this as the energy of a capacitor, since for a capacitor

$$W_E = \frac{1}{2} Q V_a$$

And the total charge is $Q = 2\pi a L \rho_s$, combined with V_a obtained previously.

To obtaine an expression for energy density in J/m^3, reconsider

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot \mathbf{E}\, dv$$

Rewrite for an elemental volume;

$$dW_E = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dv$$

From which, energy density is obtained as;

$$w_e = \frac{dW}{dv} = \frac{1}{2}\boldsymbol{D} \cdot \boldsymbol{E}$$

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