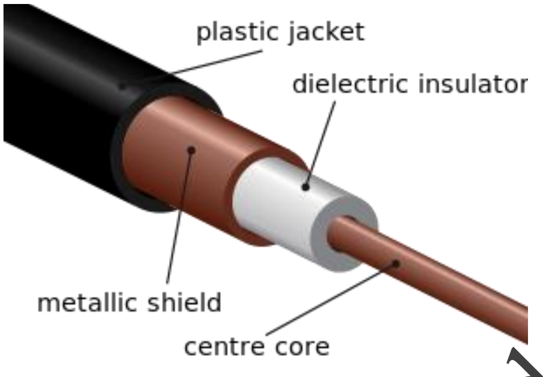
Transmission lines

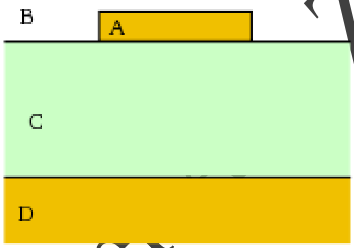
Chapter 10

TRANSMISSION LINES AND THEIR TIME DOMAIN WAVE EQUATIONS

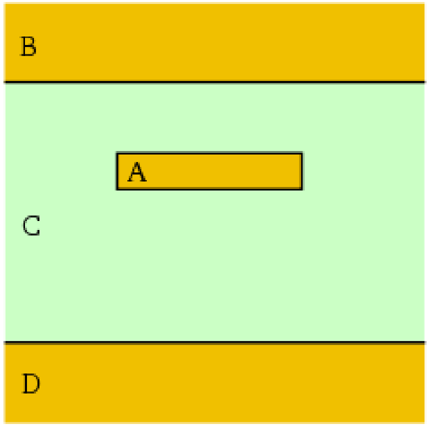
Most common types include:



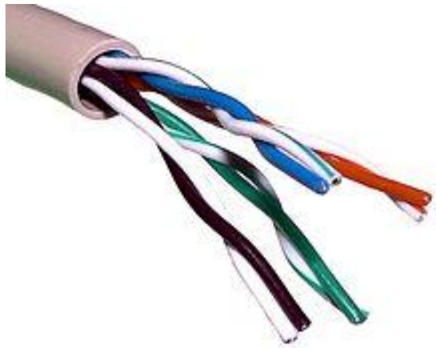
Coaxial cable



Cross-section of microstrip geometry. Conductor (A) is separated from ground plane (D) by dielectric substrate (C). Upper dielectric (B) is typically air



Cross-section diagram of stripline geometry. Central conductor (A) is sandwiched between ground planes (B and D). Structure is supported by dielectric (C



Unshielded twisted pair cable with different twist rates

- Two or more conductors surrounded by a dielectric.
- Used to transmit electric energy and information bearing signals form one point to another.
- Lossless TL implies perfect conductors and perfect dielectrics.
- Distributed parameter network.
- Voltages and currents vary spatially besides time variation.
- TEM: Transverse ElectroMagnetic.

Divide the line into small segments, and consider a differential length Δz of the line:

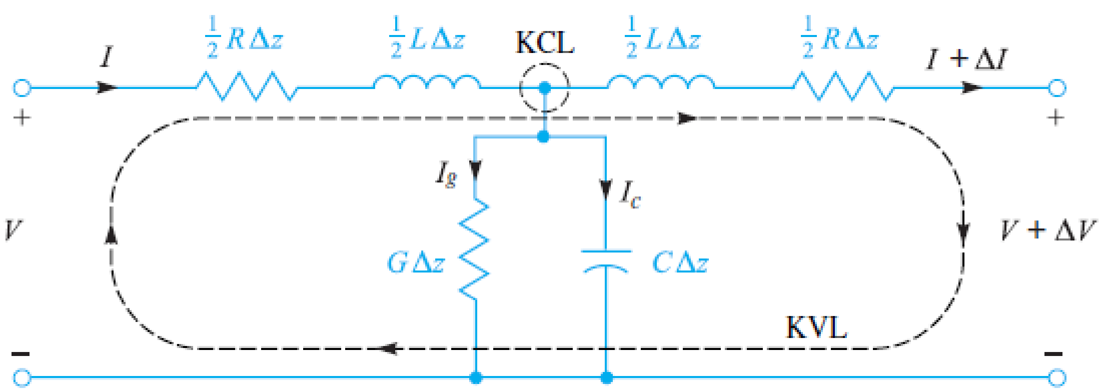


Figure 10.3 Lumped-element model of a short transmission line section with losses. The length of the section is Δz. Analysis involves applying Kirchoff's voltage and current laws (KVL and KCL) to the indicated loop and node, respectively.

$R, L, G,$ *and* C are per unit length parameters.

Parameters	Coaxial Line	Two-Wire Line	Planar Line
R (Ω/m)	$\frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a}+\frac{1}{b}\right]$ ($\delta \ll a, c-b$)	$\frac{1}{\pi a\delta\sigma_c}$ ($\delta \ll a$)	$\frac{2}{w\delta\sigma_c}$ ($\delta \ll t$)
L (H/m)	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a}$	$\frac{\mu d}{w}$
G (S/m)	$\frac{2\pi\sigma}{\ln\frac{b}{a}}$	$\frac{\pi\sigma}{\cosh^{-1}\frac{d}{2a}}$	$\frac{\sigma w}{d}$
C (F/m)	$\frac{2\pi\varepsilon}{\ln\frac{b}{a}}$	$\frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}}$	$\frac{\varepsilon w}{d}$ ($w \gg d$)

$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}}$ is the skin depth of the conductor.

For each line, the conductors are characterized by σ_c, μ_c, $\varepsilon_c = \varepsilon_o$, and the homogeneous dielectric separating the conductors is characterized by σ, μ, ε.

Application of KCL and KVL gives the general *TL equations* in time domain (or *telegraphist's equations*)

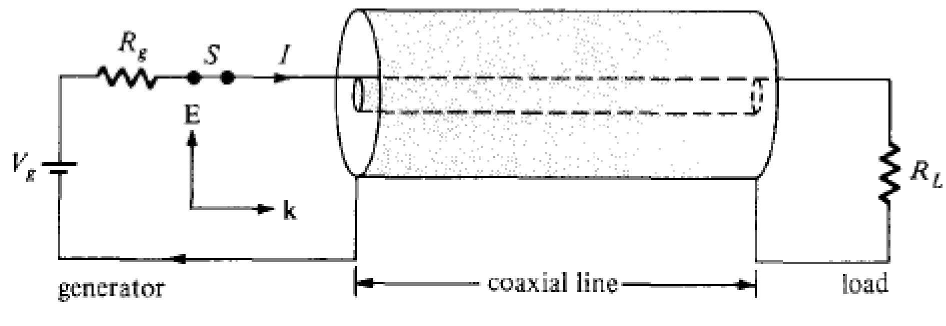
$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t}$$

$$-\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}$$

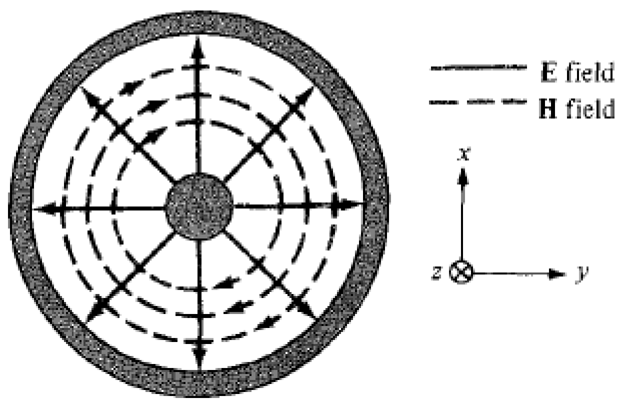
Performing some mathematics on the above equations leads to the so called TL *wave equations in time domain*

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2} + (LG+RC)\frac{\partial V(z,t)}{\partial t} + RGV(z,t)$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = LC\frac{\partial^2 I(z,t)}{\partial t^2} + (LG+RC)\frac{\partial I(z,t)}{\partial t} + RGI(z,t)$$



(a)



(b)

LOSSLESS PROPAGATION $(R = G = 0)$

- ***Lossless Line***, perfect conductors and perfect dielectrics surrounding.
- All power input to the line reaches the output.
- Voltage wave equation reduces to:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2}$$

A general solution to the above equation is assumed to be of the form:

$$V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = V^+ + V^-$$

Substituting the forward propagating part of the solution into the wave equation gives the condition:

$$v = \frac{1}{\sqrt{LC}}\ \left(m/s\right)$$

This is also clear from a dimensional check of the voltage wave equation.

HOW VOLTAGE IS RELATED TO CURRENT?

Using telegraphist equations $(R = G = 0)$, and the assumed solution for $V(z,t)$, then performing differentiation w.r.t z then integration w.r.t time, one may obtain:

$$I(z,t) = \frac{1}{Lv}\left[f_1\left(t - \frac{z}{v}\right) - f_2\left(t + \frac{z}{v}\right)\right] = I^+ + I^-$$

Identifying

$$I^+ = \frac{1}{Lv} f_1\left(t - \frac{z}{v}\right)$$

$$I^- = \frac{-1}{Lv} f_2\left(t + \frac{z}{v}\right)$$

$$Z_o = Lv = \sqrt{\frac{L}{C}} = \frac{V^+}{I^+} = -\frac{V^-}{I^-} (\Omega)$$

The ***characteristic impedance*** Z_o of the line is the ratio of positively traveling voltage wave to current wave at any point on the line.

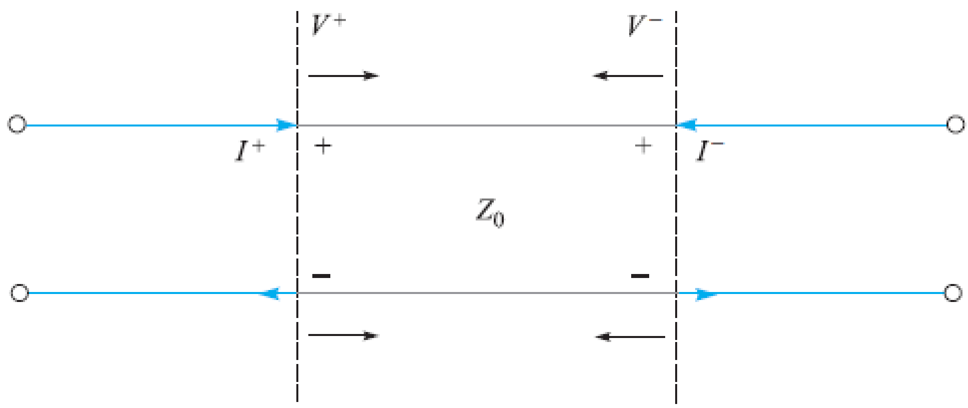


Figure 10.4 Current directions in waves having positive voltage polarity.

SINUSOIDAL VOLTAGES

Assigning for forward and backward propagating voltages a sinusoid of the form

$$V_o \cos(\omega t + \varphi)$$

Then replacing t with $t - \frac{z}{v_p}$ for forward wave and t with $t + \frac{z}{v_p}$ for backward wave:

$$V(z,t)=|V_o|\cos\left(\omega t \pm \frac{\omega}{v_p}z+\varphi\right)$$

With (assuming $\varphi=0$)

$$V_f(z,t)=|V_o|\cos\left(\omega t - \frac{\omega}{v_p}z\right)$$

$$V_b(z,t)=|V_o|\cos\left(\omega t + \frac{\omega}{v_p}z\right)$$

Define the *phase constant* as:

$$\beta = \frac{\omega}{v_p}\left(rad/m\right)$$

It represents the change in phase per metre along the path travelled by the wave at any instant

Remind yourself;

$$\omega t \rightarrow \frac{rad}{s}\times s \rightarrow rad$$

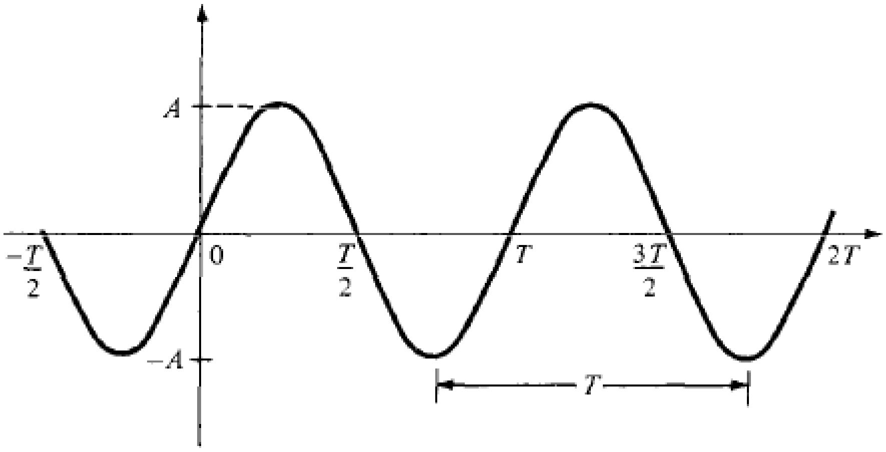
$$\beta z \rightarrow \frac{rad}{m}\times m \rightarrow rad$$

$$\beta \rightarrow \quad spatial \quad frequency$$

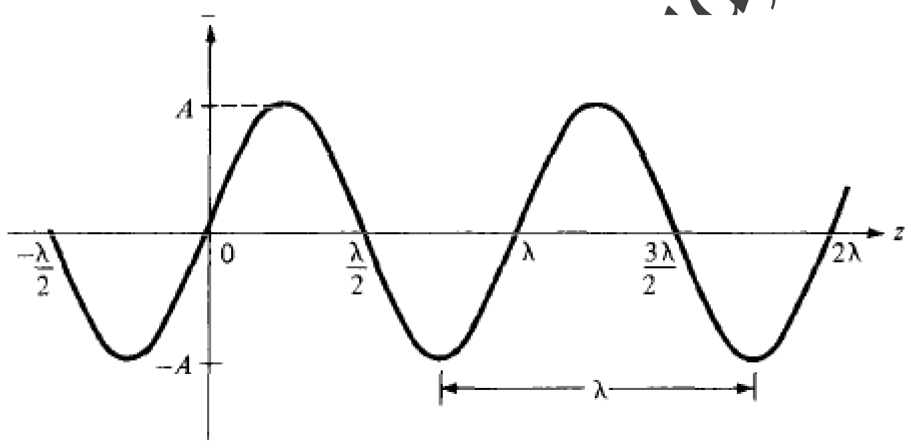
$$V_f(z,t)=|V_o|\cos(\omega t-\beta z)$$

$$V_b(z,t)=|V_o|\cos(\omega t+\beta z)$$

$$V_f(z,0)=V_b(z,0)=|V_o|\cos(\beta z)$$



$v(z,t)$ at a constant z



$v(z,t)$ at a constant time

$$\beta\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f}\,(m)$$

The wavelength of a sinusoidal wave is the spatial period of the wave—the distance over which the wave's shape repeats

TIME HARMONIC WAVES (COMPLEX ANALYSIS)

$$e^{jx} = \cos(x) + j\sin(x)$$

$$e^{-jx} = \cos(x) - j\sin(x)$$

$$\cos(x) = \mathrm{Re}\{e^{jx}\} = \frac{1}{2}e^{jx} + cc$$

$$\sin(x) = \text{Im}\{e^{jx}\} = \frac{1}{2j} e^{jx} + cc$$

Consider:

$$V_f(z,t) = |V_o| \cos(\omega t - \beta z + \varphi)$$
$$= \frac{|V_o|}{2} \left\{ e^{j(\omega t - \beta z + \varphi)} + e^{-j(\omega t - \beta z + \varphi)} \right\}$$
$$= \frac{1}{2} \left[|V_o| e^{j(\varphi)} \right] e^{-j(\beta z)} e^{j(\omega t)} + cc$$

Define:

Instantanous complex voltage

$$V_c(z,t) = V_o\, e^{\pm j\beta z} e^{j\omega t}$$

And the *phasor voltage* (dropping $e^{j\omega t}$)

$$V_s(z) = V_o\, e^{\pm j\beta z}$$

Or

$$V_f(z,t) = |V_o| \cos(\omega t - \beta z + \varphi)$$
$$= |V_o| \,\text{Re}\left\{ e^{j(\omega t - \beta z + \varphi)} \right\}$$
$$= \text{Re}\left\{ \underbrace{V_o e^{-j(\beta z)}}_{V_s(z)} e^{j\omega t} \right\}$$

To obtain time domain representation from frequency domain representation:

1. Multiply $V_s(z) = V_o\, e^{\pm j\beta z}$ by $e^{j\omega t}$.
2. Take the real part of the result.

Q: How to obtain frequency domain representation from time domain representation?

EXAMPLE 10.1

Two voltage waves having equal frequencies and amplitudes propagate in opposite directions on a lossless transmission line. Determine the total voltage as a function of time and position.

Solution. Because the waves have the same frequency, we can write their combination using their phasor forms. Assuming phase constant, β, and real amplitude, V_0, the two wave voltages combine in this way:

$$V_{sT}(z) = V_0 e^{-j\beta z} + V_0 e^{+j\beta z} = 2V_0 \cos(\beta z)$$

In real instantaneous form, this becomes

$$\mathcal{V}(z,t) = \text{Re}[2V_0 \cos(\beta z) e^{j\omega t}] = 2V_0 \cos(\beta z)\cos(\omega t)$$

We recognize this as a *standing wave,* in which the amplitude varies, as $\cos(\beta z)$, and oscillates in time, as $\cos(\omega t)$. Zeros in the amplitude (nulls) occur at fixed locations, $z_n = (m\pi)/(2\beta)$ where m is an odd integer. We extend the concept in Section 10.10, where we explore the *voltage standing wave ratio* as a measurement technique.

TL WAVE EQUATIONS AND THEIR SOLUTIONS IN PHASOR FORM

Recall:

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t}$$

$$-\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}$$

Rewriting voltages and currents in terms of their phasor representations, then performing the indicated differentiations and dropping the $e^{j\omega t}$ term, one can obtain:

$$-\frac{dV_s(z)}{dz} = (R + j\omega L)I_s(z) \quad \rightarrow (1)$$

$$-\frac{dI_s(z)}{dz} = (G + j\omega C)V_s(z) \quad \rightarrow (2)$$

To obtain the wave equations in frequency domain, differentiate (1) w.r.t. z then substitute (2) into the result.

$$\frac{d^2V_s(z)}{dz^2} = (R + j\omega L)(G + j\omega C)V_s(z)$$

$$\frac{d^2I_s(z)}{dz^2} = (R + j\omega L)(G + j\omega C)I_s(z)$$

The *propagation constant* γ is defined as:

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY}$$
$$= \alpha + j\beta$$

And the *solution to the voltage wave equation* is given by:

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

The relation between voltage and current in frequency domain is found from telegraphist equations namely:

$$-\frac{dV_s(z)}{dz} = (R + j\omega L) I_s(z) \quad \rightarrow (1)$$
$$-\frac{dI_s(z)}{dz} = (G + j\omega C) V_s(z) \quad \rightarrow (2)$$

Substituting the expressions for $V_s(z)$ and $I_s(z)$, then matching exponents, one may obtain:

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}}$$
$$= \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0| e^{j\theta}$$

EXAMPLE 10.2

A lossless transmission line is 80 cm long and operates at a frequency of 600 MHz. The line parameters are $L = 0.25\ \mu$H/m and $C = 100$ pF/m. Find the characteristic impedance, the phase constant, and the phase velocity.

Solution. Because the line is lossless, both R and G are zero. The characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = 50\ \Omega$$

Because $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC}$, we see that

$$\beta = \omega\sqrt{LC} = 2\pi(600 \times 10^6)\sqrt{(0.25 \times 10^{-6})(100 \times 10^{-12})} = 18.85 \text{ rad/m}$$

Also,

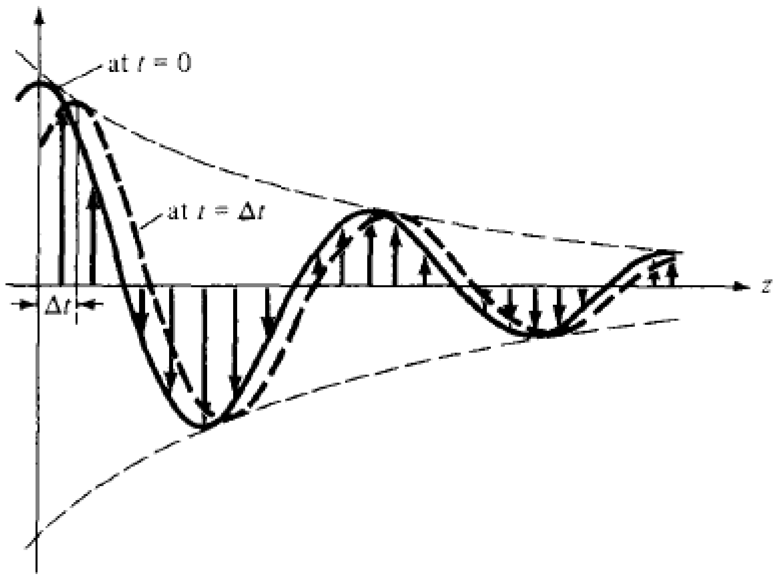
$$v_p = \frac{\omega}{\beta} = \frac{2\pi(600 \times 10^6)}{18.85} = 2 \times 10^8 \text{ m/s}$$

Reconsider:

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$V_s(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z}$$

$$V(z,t) = \underbrace{V_0^+ e^{-\alpha z} \cos(\omega t - \beta z)}_{Forward\ Wave} + \underbrace{V_0^- e^{\alpha z} \cos(\omega t + \beta z)}_{Backward\ Wave}$$



Voltage traveling along +z-direction at times $t = 0$ and $t = \Delta t$;.

LOSSLESS PROPAGATION (REVISITED)

$$Lossless \rightarrow R = G = 0 \rightarrow \gamma = j\omega\sqrt{LC}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC} \rightarrow \alpha = 0 \ \& \ \beta = \omega\sqrt{LC}$$

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$V_f(z,t) = V_0^+ \cos(\omega t - \beta z)$$

$$V_b(z,t) = V_0^- \cos(\omega t + \beta z)$$

LOW-LOSS PROPAGATION

$$\text{Low - loss} \rightarrow R << \omega L \text{ and } G << \omega C$$

Reconsider:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1 + \frac{G}{j\omega C}\right)^{\frac{1}{2}}$$

Using the first three terms in the binomial series expansion, namely:

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + ... \quad for\ x << 1$$

Then, the attenuation and propagation constants may be approximated by:

$$\alpha = \text{Re}\{\gamma\} \approx \frac{1}{2}\left[R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right]$$

$$\beta = \text{Im}\{\gamma\} \approx \omega\sqrt{LC}\left[1 + \frac{1}{8}\left(\frac{G}{\omega C} - \frac{R}{\omega L}\right)^2\right]$$

Similar argument may be applied to the characteristic impedance:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\approx \sqrt{\frac{L}{C}}\left\{1 + \frac{1}{2\omega^2}\left[\frac{1}{4}\left(\frac{R}{L} + \frac{G}{C}\right)^2 - \frac{G^2}{C^2}\right] + \frac{j}{2\omega}\left(\frac{G}{C} - \frac{R}{L}\right)\right\}$$

Note that:

- $\alpha \propto R\ and\ G$.
- β is a *non-linear function of frequency* $\beta(\omega)$, then $v_p = \frac{\omega}{\beta}$ is frequency dependent.
- The *group velocity* $v_g = \frac{d\omega}{d\beta}$ also depends on frequency $\Rightarrow$ Signal distortion.
- A constant phase and group velocities may be obtained, even when $R \neq 0\ and\ G \neq 0$. This occurs when:

$$\frac{R}{L} = \frac{G}{C}$$ (Distortionless line)

$$\alpha = \mathrm{Re}\{\gamma\} \approx R\sqrt{\frac{C}{L}}$$

$$\beta \approx \omega\sqrt{LC}\left[1+\frac{1}{8}\left(\frac{1}{\omega}\left\{\frac{G}{C}-\frac{R}{L}\right\}\right)^2\right] = \omega\sqrt{LC}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega} = \frac{1}{\sqrt{LC}}$$

$$Z_0 \approx \sqrt{\frac{L}{C}}$$

EXAMPLE 10.3

Suppose in a certain transmission line $G = 0$, but R is finite valued and satisfies the low-loss requirement, $R \ll \omega L$. Use Eq. (56) to write the approximate magnitude and phase of Z_0.

Solution. With $G = 0$, the imaginary part of (56) is much greater than the second term in the real part [proportional to $(R/\omega L)^2$]. Therefore, the characteristic impedance becomes

$$Z_0(G=0) \doteq \sqrt{\frac{L}{C}}\left(1 - j\frac{R}{2\omega L}\right) = |Z_0|e^{j\theta}$$

where $|Z_0| \doteq \sqrt{L/C}$, and $\theta = \tan^{-1}(-R/2\omega L)$.

D10.1. At an operating radian frequency of 500 Mrad/s, typical circuit values for a certain transmission line are: $R = 0.2\ \Omega$/m, $L = 0.25\ \mu$H/m, $G = 10\ \mu$S/m, and $C = 100$ pF/m. Find: (*a*) α; (*b*) β; (*c*) λ; (*d*) v_p; (*e*) Z_0.

Ans. 2.25 mNp/m; 2.50 rad/m; 2.51 m; 2×10^8 m/sec; $50.0 - j0.0350\ \Omega$

POWER TRANSMISSION AND LOSS

$$V_s(z) = V_0^+ e^{-\alpha z}e^{-j\beta z} + V_0^- e^{\alpha z}e^{j\beta z}$$

$$V_s(z) = |V_0^+|e^{j\theta^+}e^{-\alpha z}e^{-j\beta z} + |V_0^-|e^{j\theta^-}e^{\alpha z}e^{j\beta z}$$

$$I_s(z) = |I_0^+| e^{j\varphi^+} e^{-\alpha z} e^{-j\beta z} + |I_0^-| e^{j\varphi^-} e^{\alpha z} e^{j\beta z}$$

And since

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{|V_o^+|}{|I_o^+|} e^{j(\theta^+ - \varphi^+)} = |Z_0| e^{j\theta_{Z_o}}$$

Then

$$I_s(z) = \frac{V_0^+}{Z_o} e^{-\alpha z} e^{-j\beta z} - \frac{V_0^-}{Z_o} e^{\alpha z} e^{j\beta z}$$

Considering the forward waves

$$V_{sf}(z) = |V_0^+| e^{j\theta^+} e^{-\alpha z} e^{-j\beta z}$$

$$I_{sf}(z) = \frac{|V_0^+|}{|Z_o|} e^{j\varphi^+} e^{-\alpha z} e^{-j\beta z}$$

The *Instantaneous power* $p(z,t)$ is defined as:

$$p(z,t) = V_f(z,t) I_f(z,t)$$

Is evaluated to give

$$p(z,t) = \frac{|V_o^+|^2}{Z_o} e^{-2\alpha z} \cos(\omega t - \beta z + \theta^+) \cos(\omega t - \beta z + \varphi^+)$$

And the *time-averaged power* is given by:

$$\langle p \rangle = \frac{1}{T} \int_T p(z,t) dt$$

This may be evaluated to give:

$$\langle p \rangle = \frac{|V_o^+|^2}{2Z_o} e^{-2\alpha z} \cos(\theta_{Z_o})$$

The same result may be obtained more easily if the <u>*average power is defined as*</u>:

$$\langle p \rangle = \frac{1}{2} \mathrm{Re}\{ V_s(z)\, I_s^*(z) \}$$

This may be evaluated to give:

$$\langle p \rangle = \frac{1}{2} \mathrm{Re}\left[\left(|V_0^+| e^{j\theta^+} e^{-\alpha z} e^{-j\beta z} \right) \left(\frac{|V_0^+|}{|Z_o|} e^{-j\varphi^+} e^{-\alpha z} e^{j\beta z} \right) \right]$$

$$\langle p \rangle = \frac{\left|V_o^+\right|^2}{2Z_o} e^{-2\alpha z} \cos\left(\theta_{Z_o}\right)$$

As a measure of power drop along a lossy line, consider:

$$\langle p(0) \rangle = \frac{\left|V_o^+\right|^2}{2Z_o} \cos\left(\theta_{Z_o}\right)$$

So

$$\langle p(z) \rangle = \langle p(0) \rangle e^{-2\alpha z}$$

Then

$$\frac{\langle p(0) \rangle}{\langle p(z) \rangle} = e^{2\alpha z}$$

In dB

$$\text{Power Loss in dB} = 10\text{Log}_{10}\left[\frac{\langle p(0) \rangle}{\langle p(z) \rangle}\right] = 8.69\alpha z$$

EXAMPLE 10.4

A 20-m length of transmission line is known to produce a 2.0-dB drop in power from end to end. (*a*) What fraction of the input power reaches the output? (*b*) What fraction of the input power reaches the midpoint of the line? (*c*) What exponential attenuation coefficient, α, does this represent?

Solution. (*a*) The power fraction will be

$$\frac{\langle \mathcal{P}(20) \rangle}{\langle \mathcal{P}(0) \rangle} = 10^{-0.2} = 0.63$$

(*b*) 2 dB in 20 m implies a loss rating of 0.2 dB/m. So, over a 10-m span, the loss is 1.0 dB. This represents the power fraction, $10^{-0.1} = 0.79$.

(*c*) The exponential attenuation coefficient is found through

$$\alpha = \frac{2.0 \text{ dB}}{(8.69 \text{ dB/Np})(20 \text{ m})} = 0.012 \text{ [Np/m]}$$

A final point addresses the question: Why use decibels? The most compelling reason is that when evaluating the accumulated loss for several lines and devices that are all end-to-end connected, the net loss in dB for the entire span is just the sum of the dB losses of the individual elements.

WAVE REFLECTIONS @ DISCONTINUITIES

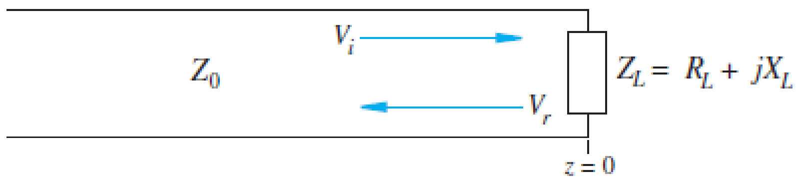


Figure 10.5 Voltage wave reflection from a complex load impedance.

$$V_i(z) = V_{oi}\, e^{-\alpha z} e^{-j\beta z}$$
$$V_r(z) = V_{or}\, e^{\alpha z} e^{j\beta z}$$

At $z = 0$ the load voltage and the load current are:

$$V_L = V_{oi} + V_{or}$$

$$I_L = \frac{V_L}{Z_L}$$

$$I_L = I_{oi} + I_{or} = \frac{1}{Z_o}\left[V_{oi} - V_{or}\right]$$

So

$$I_L = \frac{1}{Z_o}\left[V_{oi} - V_{or}\right] = \frac{V_L}{Z_L} = \frac{1}{Z_L}\left[V_{oi} + V_{or}\right]$$

Rearranging and solving for the ratio $\frac{V_{or}}{V_{oi}}$

$$\frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_o}{Z_L + Z_o}$$

The voltage ***reflection coefficient*** at the load is defined as:

$$\Gamma_L = \frac{V_{or}}{V_{oi}} = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma_L| e^{j\theta_\Gamma}$$

Also since

$$V_L = V_{oi} + V_{or} = V_{oi} + \Gamma V_{oi}$$

Solve for the so defined voltage ***transmission coefficient***:

$$\tau_L = \frac{V_L}{V_{oi}} = 1 + \Gamma = \frac{2Z_L}{Z_L + Z_o} = |\tau_L| e^{j\theta_\tau}$$

- What is the condition required for the load to receive all transmitted power (all power input to line)?

$$\Gamma_L = 0 = \frac{Z_L - Z_o}{Z_L + Z_o} \Rightarrow Z_L = Z_o$$

Load matched to line when

$$Z_L = Z_o$$

- What fractions of incident power are reflected and dissipated by the load? The load in this case is assumed to be located at $z = L$.

$$\langle p_i(z)\rangle\Big|_{z=L} = \frac{|V_o^+|^2}{2Z_o} e^{-2\alpha z}\cos(\theta_{Z_o})\Big|_{z=L}$$

$$= \frac{|V_o^+|^2}{2Z_o} e^{-2\alpha L}\cos(\theta_{Z_o})$$

Also the power reflected from the load is:

$$\langle p_r\rangle\Big|_{z=L} = \frac{|\Gamma|^2|V_o^+|^2}{2Z_o} e^{-2\alpha L}\cos(\theta_{Z_o})$$

(It would be a good exercise for you to derive the above result)

So, from the above we have:

$$\frac{\langle p_r\rangle}{\langle p_i\rangle} = |\Gamma|^2 = \Gamma\Gamma^*$$

And

$$\frac{\langle p_t\rangle}{\langle p_i\rangle} = 1 - |\Gamma|^2$$

For a wave incident from a *semi-infinite* TL to a second semi-infinite TL, the second may be treated as a load

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

EXAMPLE 10.5

A 50-Ω lossless transmission line is terminated by a load impedance, $Z_L = 50 - j75\ \Omega$. If the incident power is 100 mW, find the power dissipated by the load.

Solution. The reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j75 - 50}{50 - j75 + 50} = 0.36 - j0.48 = 0.60e^{-j.93}$$

Then

$$\langle \mathcal{P}_t\rangle = (1 - |\Gamma|^2)\langle \mathcal{P}_i\rangle = [1 - (0.60)^2](100) = 64 \text{ mW}$$

EXAMPLE 10.6

Two lossy lines are to be joined end to end. The first line is 10 m long and has a loss rating of 0.20 dB/m. The second line is 15 m long and has a loss rating of 0.10 dB/m. The reflection coefficient at the junction (line 1 to line 2) is $\Gamma = 0.30$. The input power (to line 1) is 100 mW. (*a*) Determine the total loss of the combination in dB. (*b*) Determine the power transmitted to the output end of line 2.

Solution. (*a*) The dB loss of the joint is

$$L_j(\text{dB}) = 10\log_{10}\left(\frac{1}{1-|\Gamma|^2}\right) = 10\log_{10}\left(\frac{1}{1-0.09}\right) = 0.41 \text{ dB}$$

The total loss of the link in dB is now

$$L_t(\text{dB}) = (0.20)(10) + 0.41 + (0.10)(15) = 3.91 \text{ dB}$$

(*b*) The output power will be $P_{out} = 100 \times 10^{-0.391} = 41$ mW.

VSWR

(Measure of the degree of mismatch of a TL)

$$VSWR = s = \frac{|V_s(z)|_{\max}}{|V_s(z)|_{\min}}$$

Assuming lossless line, and starting with

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

Then

$$V_s(z) = V_0^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right),$$

$$\text{since } \Gamma_L = \frac{V_0^-}{V_0^+} = |\Gamma_L| e^{j\theta_\Gamma}$$

The above expression may be written as:

$$V_s(z) = V_0\left(1 - |\Gamma_L|\right)e^{-j\beta z} + 2V_0|\Gamma_L| e^{j\frac{\theta_\Gamma}{2}} \cos\left(\beta Z + \frac{\theta_\Gamma}{2}\right)$$

$$\text{where } V_0 = V_0^+$$

Transferring into time domain

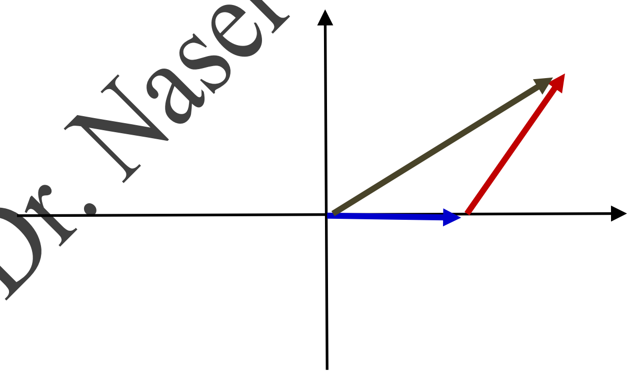
$$V(z,t) = \underbrace{V_0\left(1-|\Gamma_L|\right)\cos(\omega t - \beta z)}_{\text{Traveling part}}$$

$$+\underbrace{2V_0|\Gamma_L|\cos\left(\beta Z + \frac{\theta_\Gamma}{2}\right)\cos\left(\omega t + \frac{\theta_\Gamma}{2}\right)}_{\text{Standing part}}$$

Portion of the first incident wave reflects back and propagates in the line, and *interferes* with an equivalent portion of the 2nd incident wave to form a standing wave, the rest of the incident wave (which does not interfere) is the traveling wave part.

WHAT IS THE VOLTAGE MAXIMUM AND MINIMUM AND WHERE DO THEY OCCUR?

$$\begin{aligned} V_s(z) &= V_0\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right) \\ &= V_0\left(e^{-j\beta z} + |\Gamma_L| e^{j(\beta z + \theta_\Gamma)}\right) \\ &= V_0\, e^{-j\beta z}\left(1 + |\Gamma_L| e^{j(2\beta z + \theta_\Gamma)}\right) \end{aligned}$$



Maximum's occur when:

$$2\beta z + \theta_\Gamma = -2m\pi, \quad m = 0,1,2,....$$

Hence;

$$z_{\max} = \frac{-1}{2\beta}\left(2m\pi + \theta_\Gamma\right)$$

Then

$$\begin{aligned} \left|V_s(z)\right|_{\max} &= \left|V_0\, e^{-j\beta z}\left(1 + |\Gamma_L| e^{j(2\beta z + \theta_\Gamma)}\right)\right|_{z=z_{\max}} \\ &= V_0\left(1 + |\Gamma_L|\right) \end{aligned}$$

Minimum's occur when:

$$2\beta z+\theta_\Gamma=-(2m+1)\pi,\quad m=0,1,2,\ldots.$$

So;

$$z_{\min}=\frac{-1}{2\beta}\left([2m+1]\pi+\theta_\Gamma\right)$$

Then

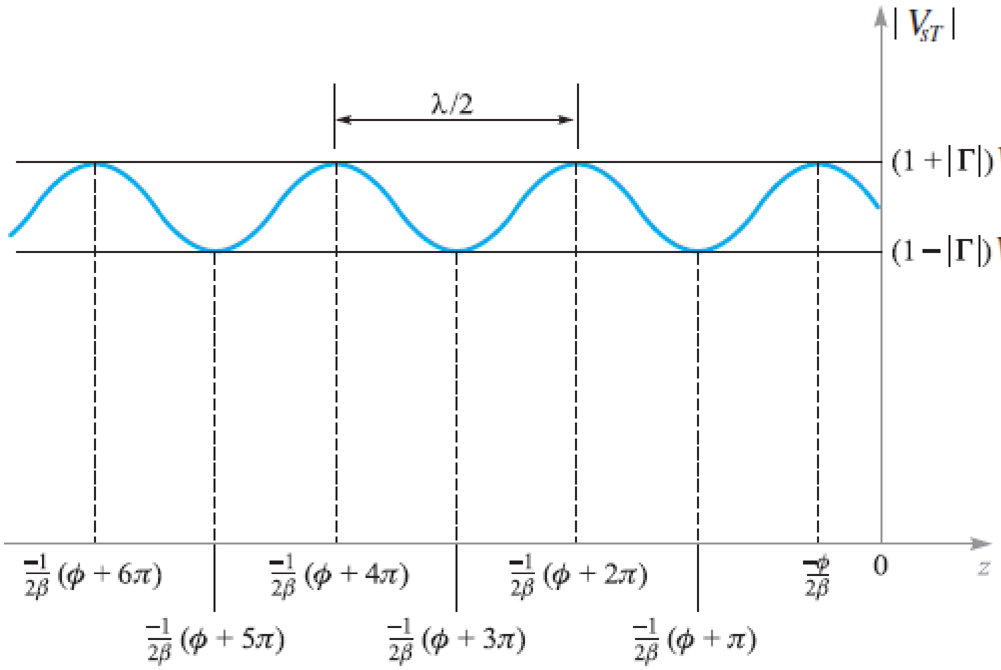
$$\left|V_s(z)\right|_{\min}=\left|V_0\,e^{-j\beta z}\left(1+|\Gamma_L|e^{j(2\beta z+\theta_\Gamma)}\right)\right|_{z=z_{\min}}$$

$$=V_0\left(1-|\Gamma_L|\right)$$

And the VSWR is obtained easily as:

$$VSWR=s=\frac{|V_s(z)|_{\max}}{|V_s(z)|_{\min}}=\frac{1+|\Gamma|}{1-|\Gamma|}$$

$$|\Gamma|=\frac{s-1}{s+1}$$



Plot of the magnitude of V_{sT} as found from $V_{sT}(z)=V_0\left(e^{-j\beta z}+|\Gamma_L|e^{j(\beta z+\theta_\Gamma)}\right)$ as a function of position, z, in front of the load (at z = 0). The reflection coefficient phase is θ_Γ, which leads to the indicated locations of maximum and minimum voltage amplitude, as found from $z_{\min}=\frac{-1}{2\beta}\left([2m+1]\pi+\theta_\Gamma\right)$ and $z_{\max}=\frac{-1}{2\beta}\left(2m\pi+\theta_\Gamma\right)$.

Implication: $|\Gamma|$ maybe found from measured s, and θ_Γ may be found from measured locations of maximum's and minimum's. Then the load impedance is known.

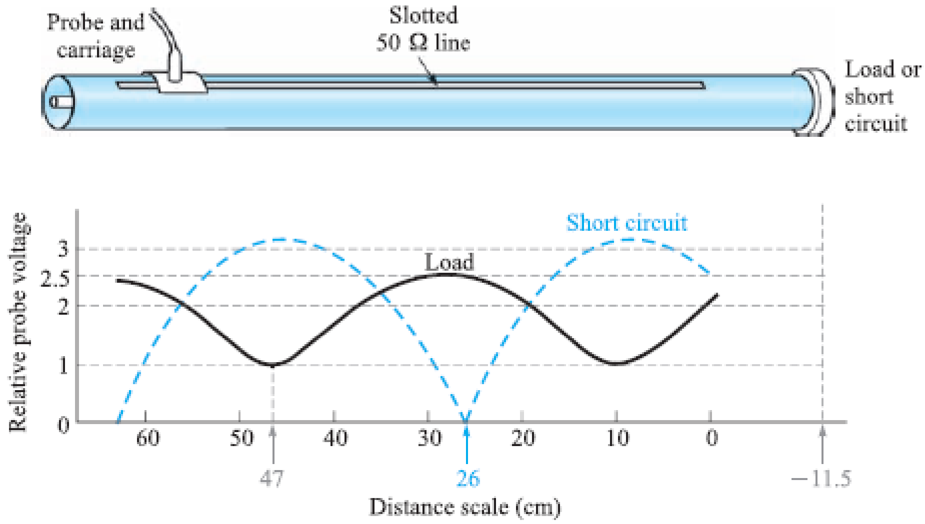


Figure 10.15 A sketch of a coaxial slotted line. The distance scale is on the slotted line. With the load in place, $s = 2.5$, and the minimum occurs at a scale reading of 47 cm. For a short circuit, the minimum is located at a scale reading of 26 cm. The wavelength is 75 cm.

EXAMPLE 10.7

Slotted line measurements yield a VSWR of 5, a 15-cm spacing between successive voltage maxima, and the first maximum at a distance of 7.5 cm in front of the load. Determine the load impedance, assuming a 50-Ω impedance for the slotted line.

Solution. The 15-cm spacing between maxima is $\lambda/2$, implying a wavelength of 30 cm. Because the slotted line is air-filled, the frequency is $f = c/\lambda = 1$ GHz. The first maximum at 7.5 cm is thus at a distance of $\lambda/4$ from the load, which means that a voltage minimum occurs at the load. Thus Γ will be real and negative. We use (92) to write

$$|\Gamma| = \frac{s-1}{s+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

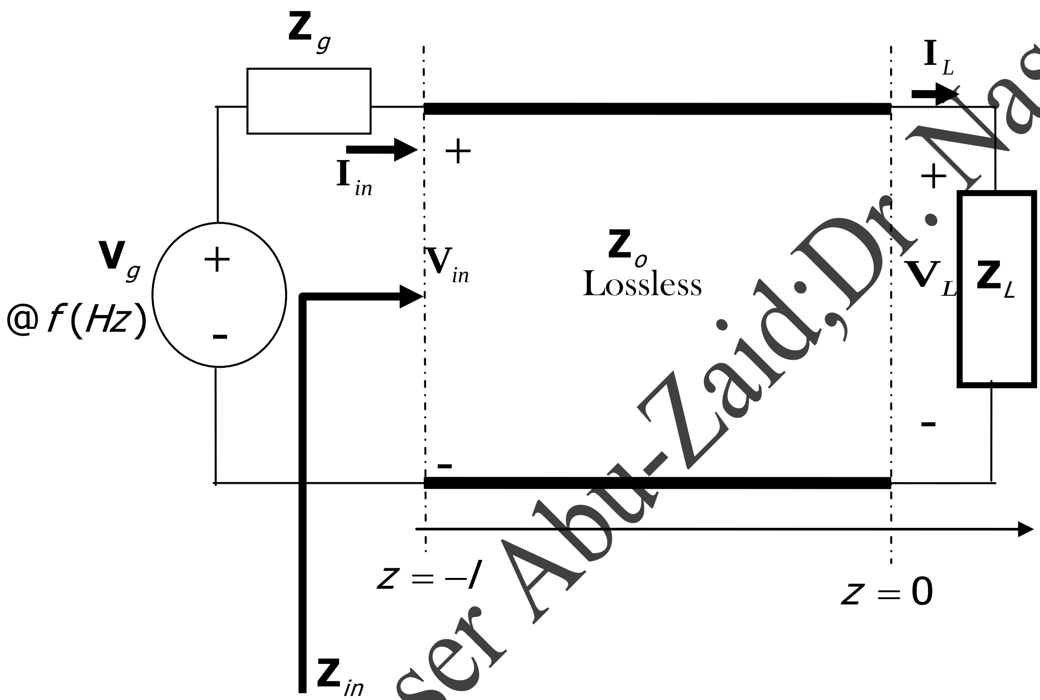
So

$$\Gamma = -\frac{2}{3} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

which we solve for Z_L to obtain

$$Z_L = \frac{1}{5} Z_0 = \frac{50}{5} = 10\ \Omega$$

FINITE LENGTH TL'S (TL CIRCUITS)



$$Z_{in} = ? \quad \text{and} \quad \Gamma(z) = ?$$

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I_s(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}$$

$$\Gamma_L = \frac{V_0^-}{V_0^+} = |\Gamma_L| e^{j\theta_\Gamma}, \quad I_0^+ = \frac{V_0^+}{Z_0}, \quad I_0^- = \frac{-V_0^-}{Z_0}$$

Define the *wave impedance* $Z_w(z)$ at any z as:

$$Z_w(z) = \frac{V_s(z)}{I_s(z)}$$

$$Z_w(z) = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}}$$

$$Z_w(z) = \frac{V_0^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)}{\frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma_L e^{j\beta z}\right)}$$

$$Z_w(z) = Z_0 \frac{\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)}{\left(e^{-j\beta z} - \Gamma_L e^{j\beta z}\right)}$$

Using Euler's identity and the fact that $\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$. Then If evaluated at $z = -l$

$$Z_w(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

$$Z_{in} = Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)}$$

Also a *generalized reflection coefficient* may be defined as follows:

$$\Gamma(z) = \frac{V_0^- e^{j\beta z}}{V_0^+ e^{-j\beta z}} = \frac{V_0^-}{V_0^+} e^{j2\beta z}$$

$$= \Gamma_L e^{j2\beta z} = |\Gamma_L| e^{j(\theta_\Gamma + 2\beta z)}$$

$$\Gamma(z) = |\Gamma_L| e^{j(\theta_\Gamma + 2\beta z)}$$

$$\Gamma(0) = |\Gamma_L| e^{j(\theta_\Gamma + 2\beta[0])} = |\Gamma_L| e^{j(\theta_\Gamma)} = \Gamma_L$$

$$\Gamma(-l) = |\Gamma_L| e^{j(\theta_\Gamma - 2\beta l)}$$

Also, note that the *wave impedance* may be obtained as:

$$Z_w(z) = Z_0 \frac{\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)}{\left(e^{-j\beta z} - \Gamma_L e^{j\beta z}\right)}$$

$$= Z_0 \frac{\left(1 + |\Gamma_L| e^{j(\theta_\Gamma + 2\beta z)}\right)}{\left(1 - |\Gamma_L| e^{j(\theta_\Gamma + 2\beta z)}\right)}$$

$$= Z_0 \frac{(1 + \Gamma(z))}{(1 - \Gamma(z))}$$

And @ $z=-l$, the input impedance becomes;

$$Z_{in}=Z_w(-l)=Z_0\frac{\left(1+|\Gamma_L|e^{j(\theta_\Gamma-2\beta l)}\right)}{\left(1-|\Gamma_L|e^{j(\theta_\Gamma-2\beta l)}\right)}$$

Note also that:

$$V_s(0)=V_0^+[1+\Gamma_L]$$

So

$$V_0^+=\frac{V_s(0)}{[1+\Gamma_L]}$$

Special cases:

1) Half wave length line:

$$l=m\frac{\lambda}{2},\ m=1,2,3,\dots\dots.$$

$$\beta l=m\pi$$

$$Z_{in}=Z_0\frac{Z_L\cos(m\pi)+jZ_0\sin(m\pi)}{Z_0\cos(m\pi)+jZ_L\sin(m\pi)}$$

$$Z_{in}=Z_L$$

2) Quarter wave transformer:

$$l=(2m+1)\frac{\lambda}{4},\ m=0,1,2,3,\dots\dots.$$

Odd multiples of $\frac{\lambda}{4}$

$$\beta l=(2m+1)\frac{\pi}{2}$$

$$Z_{in}=Z_0\frac{Z_L\cos\left([2m+1]\frac{\pi}{2}\right)+jZ_0\sin\left([2m+1]\frac{\pi}{2}\right)}{Z_0\cos\left([2m+1]\frac{\pi}{2}\right)+jZ_L\sin\left([2m+1]\frac{\pi}{2}\right)}$$

$$Z_{in}=\frac{Z_0^2}{Z_L}$$

The problem of joining two lines having different characteristic impedances. Suppose the impedances are (from left to right) Z_{01} and Z_{03}. At the joint, we may insert an additional line whose characteristic impedance is Z_{02} and whose length is $\lambda/4$. We thus have a sequence of joined lines whose impedances progress as Z_{01}, Z_{02}, and Z_{03}, in that order. A voltage wave is now incident from line 1 onto the joint between Z_{01} and Z_{02}. Now the effective load at the far end of line 2 is Z_{03}. The input impedance to line 2 at any frequency is now

$$Z_{in}(line\ 2) = \frac{Z_{02}^2}{Z_{03}}$$

Reflections at the $Z_{01}-Z_{02}$ interface will not occur if $Z_{in} = Z_{01}$. Therefore, we can match the junction (allowing complete transmission through the three-line sequence) if Z_{02} is chosen so that

$$Z_{02} = \sqrt{Z_{01}Z_{03}}$$

This technique is called *quarter-wave matching*.

3) Short Circuit termination:

$$Z_L = 0$$

$$Z_{in} = Z_0 \frac{(0)\cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j(0)\sin(\beta l)}$$

$$Z_{in_{sc}} = jZ_0 \tan(\beta l)$$

4) Open circuit termination:

$$Z_L \to \infty$$

$$Z_{in} = \lim_{Z_L \to \infty} Z_0 \frac{\cos(\beta l) + j\frac{Z_0 \sin(\beta l)}{Z_L}}{\frac{Z_0 \cos(\beta l)}{Z_L} + j\sin(\beta l)}$$

$$Z_{in_{oc}} = -jZ_0 \cot(\beta l)$$

Note also that Z_0 may be found from measurements of short and open circuit terminations

$$Z_0 = \sqrt{Z_{in_{oc}} Z_{in_{sc}}}$$

Example:

1) Calculate the load reflection coefficient, the standing wave ratio, the wavelength on the line, the phase constant, the attenuation constant, the electrical length of the line, the input impedance offered to the source, the voltage at the input to the line, the time domain input voltage, the time domain load voltage, the time domain input current, the time domain load current, the average power delivered to the input of the line, the average power delivered to the load by the line.
2) If a 300Ω load is connected in parallel with the first load then calculate: the reflection coefficient, the standing wave ratio, the input impedance offered to the source, the phasor input current, the average power supplied to the line by the source, the average power received by each load, the phasor voltage across each load, where is the voltage maximums and minimums and what are those values, the phasor load voltage.

1) The line is matched;
The reflection coefficient is zero; $\Gamma_L = 0$
The standing wave ratio is unity; $s = 1$

$$\lambda = \frac{v}{f} = 2.5(m)$$

$$\beta = \frac{2\pi}{\lambda} = 0.8\pi\left({}^{rad}/_{m}\right)$$

$$\alpha = 0$$

$$\beta l = 1.6\pi\ rad = 288^o \text{ or } l = 0.8\lambda$$

$$Z_{in} = 300(\Omega) \text{ offered to the voltage source}$$

$$V_{in} = \frac{300}{300+300}60 = 30(V)$$

The source is matched to the line and delivers the maximum available power to the line.

A transmission line that is matched at both ends produces no reflections and thus delivers maximum power to the load.

No reflection and no attenuation;

$$V_{in} = V_s(-l) = V_0^+ e^{j\beta l} + \underbrace{V_0^- e^{-j\beta l}}_{0} = 30$$

$$V_0^+ = 30 e^{-j\beta l} = 30\angle -1.6\pi$$

$$V_L = V(0) = 30\angle -1.6\pi\ rad$$

$$Vin = 30\,cos(2\pi 10^8 t)\ V$$

$$V_L = 30\,cos(2\pi 10^8 t - 1.6\pi)\ V$$

$$Iin = \frac{V_{in}}{Z_{in}} = 0.1\,cos(2\pi 10^8 t)\ A$$

$$I_L = 0.1\,cos(2\pi 10^8 t - 1.6\pi)\ A$$

The average power delivered to the input of the line by the source must all be delivered to the load by the line,

$$P_{in} = P_L = \frac{1}{2}Re\{V_{in}I_{in}^*\}$$

$$P_{in} = P_L = \frac{1}{2} \times 30 \times 0.1 = 1.5\text{ W}$$

2) • 300Ω across the line in parallel with the first receiver. The load impedance is 150Ω.

The reflection coefficient is

$$\Gamma = \frac{150-300}{150+300} = -\frac{1}{3}$$

The standing wave ratio on the line is

$$s = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$$

The input impedance is

$$Z_{in} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} = 300\frac{150\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j150\sin 288^\circ}$$

$$= 510\angle -23.8^\circ = 466 - j206\ \Omega$$

which is a capacitive impedance.

The input current phasor is

$$I_{s,in} = \frac{60}{300+466-j206} = 0.0756\angle 15.0^\circ\ \text{A}$$

The power supplied to the line by the source is

$$P_{in} = \frac{1}{2}|I_{in}|^2 R_{in}$$

$$P_{in} = \frac{1}{2} \times (0.0756)^2 \times 466 = 1.333\ \text{W}$$

Since there are no losses in the line, 1.333 W must also be delivered to the load.

This power must divide equally between two receivers, and thus each receiver now receives only 0.667W.

Because the input impedance of each receiver is 300Ω, the voltage across the receiver is easily found as

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$V_s(z) = V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)$$

$$V_{in} = V_s(-l) = V_0^+ \left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right) = 38.5\angle -8.8^o$$

$$V_0^+ = \frac{V_{in}}{e^{j\beta l} + \Gamma_L e^{-j\beta l}} = 30\angle 72^o$$

Then

$$V_L = V_s(0) = V_0^+ \left(e^{-j0} + \Gamma_L e^0\right) = 20\angle -288^o$$

The magnitude alone can be found from the power as

$$0.667 = \frac{1}{2}\frac{|V_{s,L}|^2}{300}$$

$$|V_{s,L}| = 20 \text{ V}$$

The voltage maxima is located at:

$$z_{max} = -\frac{1}{2\beta}(\phi + 2m\pi) \quad (m = 0, 1, 2, \ldots)$$

with $\beta = 0.8\pi$ and $\varphi = \pi$,

$$z_{max} = -0.625 \quad \text{and} \quad -1.875 \text{ m}$$

The minima are $\lambda/4$ distant from the maxima;

$$z_{min} = 0 \quad \text{and} \quad -1.25 \text{ m}$$

The load voltage (at *z* = 0) is a voltage minimum.

a voltage minimum occurs at the load if $Z_L < Z_o$, and a voltage maximum occurs if $Z_L > Z_o$, where both impedances are pure resistances.

EXAMPLE 10.8

In order to provide a slightly more complicated example, let us now place a purely capacitive impedance of $-j300\,\Omega$ in parallel with the two $300\,\Omega$ receivers. We are to find the input impedance and the power delivered to each receiver.

Solution. The load impedance is now $150\,\Omega$ in parallel with $-j300\,\Omega$, or

$$Z_L = \frac{150(-j300)}{150 - j300} = \frac{-j300}{1 - j2} = 120 - j60\,\Omega$$

We first calculate the reflection coefficient and the VSWR:

$$\Gamma = \frac{120 - j60 - 300}{120 - j60 + 300} = \frac{-180 - j60}{420 - j60} = 0.447\angle{-153.4^\circ}$$

$$s = \frac{1 + 0.447}{1 - 0.447} = 2.62$$

Thus, the VSWR is higher and the mismatch is therefore worse. Let us next calculate the input impedance. The electrical length of the line is still 288°, so that

$$Z_{in} = 300\,\frac{(120 - j60)\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(120 - j60)\sin 288^\circ} = 755 - j138.5\,\Omega$$

This leads to a source current of

$$I_{s,in} = \frac{V_{Th}}{Z_{Th} + Z_{in}} = \frac{60}{300 + 755 - j138.5} = 0.0564\angle 7.47^\circ\text{ A}$$

Therefore, the average power delivered to the input of the line is $P_{in} = \frac{1}{2}(0.0564)^2(755) = 1.200$ W. Since the line is lossless, it follows that $P_L = 1.200$ W, and each receiver gets only 0.6 W.

EXAMPLE 10.9

As a final example, let us terminate our line with a purely capacitive impedance, $Z_L = -j300\,\Omega$. We seek the reflection coefficient, the VSWR, and the power delivered to the load.

Solution. Obviously, we cannot deliver any average power to the load since it is a pure reactance. As a consequence, the reflection coefficient is

$$\Gamma = \frac{-j300 - 300}{-j300 + 300} = -j1 = 1\angle{-90^\circ}$$

and the reflected wave is equal in amplitude to the incident wave. Hence, it should not surprise us to see that the VSWR is

$$s = \frac{1 + |-j1|}{1 - |-j1|} = \infty$$

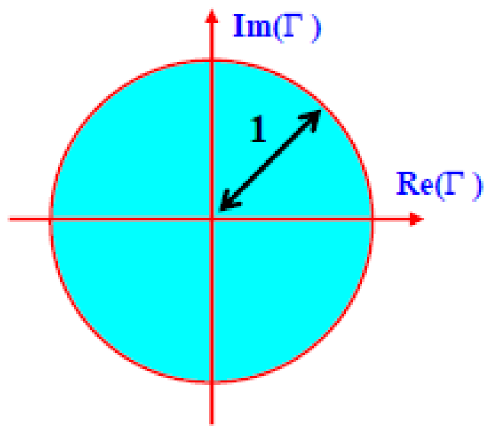
and the input impedance is a pure reactance,

$$Z_{in} = 300\,\frac{-j300\cos 288^\circ + j300\sin 288^\circ}{300\cos 288^\circ + j(-j300)\sin 288^\circ} = j589$$

Thus, no average power can be delivered to the input impedance by the source, and therefore no average power can be delivered to the load.

SMITH CHART

- The Smith chart is a graphical tool for high frequency circuit applications.
- The domain of definition of the reflection coefficient for a loss-less line is a circle of unitary radius in the complex plane. This is also the domain of the Smith chart.



The goal of the Smith chart is to identify all possible impedances on the domain of existence of the reflection coefficient. To do so, we start from the general definition of line impedance (which is equally applicable to a load impedance when d=0)

$$Z(d) = \frac{V(d)}{I(d)} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

In order to obtain universal curves, we introduce the concept of normalized impedance

$$z_n(d) = \frac{Z(d)}{Z_0} = \frac{1+\Gamma(d)}{1-\Gamma(d)}$$

The normalized impedance is represented on the Smith chart by using families of curves that identify the normalized resistance *r* (real part) and the normalized reactance *x* (imaginary part)

$$z_n(d) = \text{Re}(z_n) + j\,\text{Im}(z_n) = r + jx$$

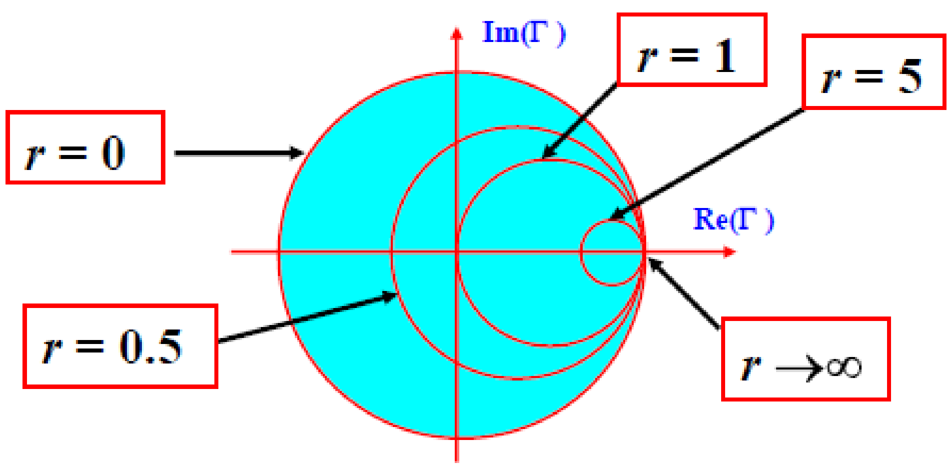
Let's represent the reflection coefficient in terms of its coordinates

$$\Gamma(\boldsymbol{d}) = \boldsymbol{Re}(\Gamma) + \boldsymbol{j}\,\boldsymbol{Im}(\Gamma)$$

After some lengthy mathematicl manipulations (follow your text book), it may by shown that the result for the real part indicates that on the complex plane with coordinates $(\boldsymbol{Re}(\boldsymbol{\Gamma}), \boldsymbol{Im}(\boldsymbol{\Gamma}))$ all the possible impedances with a given normalized resistance r are found on a circle with

$$\textbf{Center} = \left\{ \frac{r}{1+r}, 0 \right\} \qquad \textbf{Radius} = \frac{1}{1+r}$$

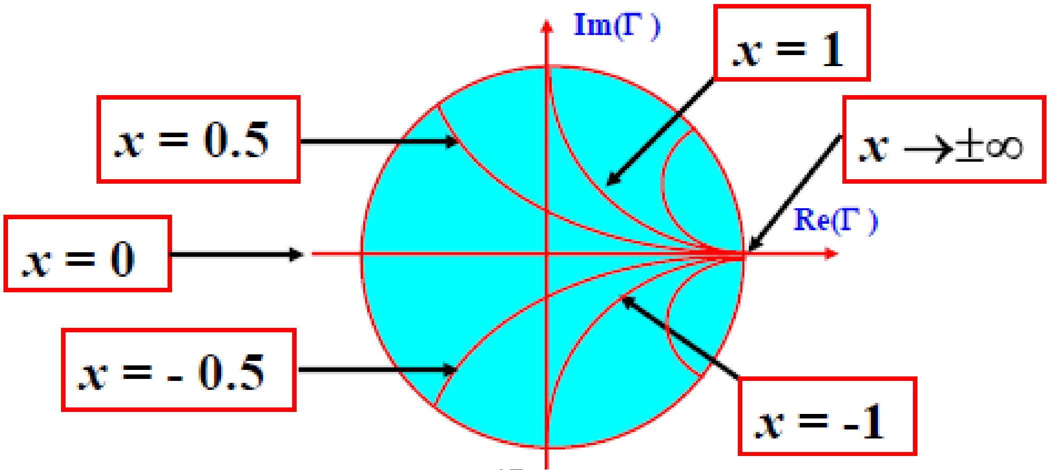
As the normalized resistance r varies from 0 to ∞, we obtain a family of circles completely contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



Also the result for the imaginary part indicates that on the complex plane with coordinates $(\boldsymbol{Re}(\Gamma), \boldsymbol{Im}(\Gamma))$ all the possible impedances with a given normalized reactance x are found on a circle with

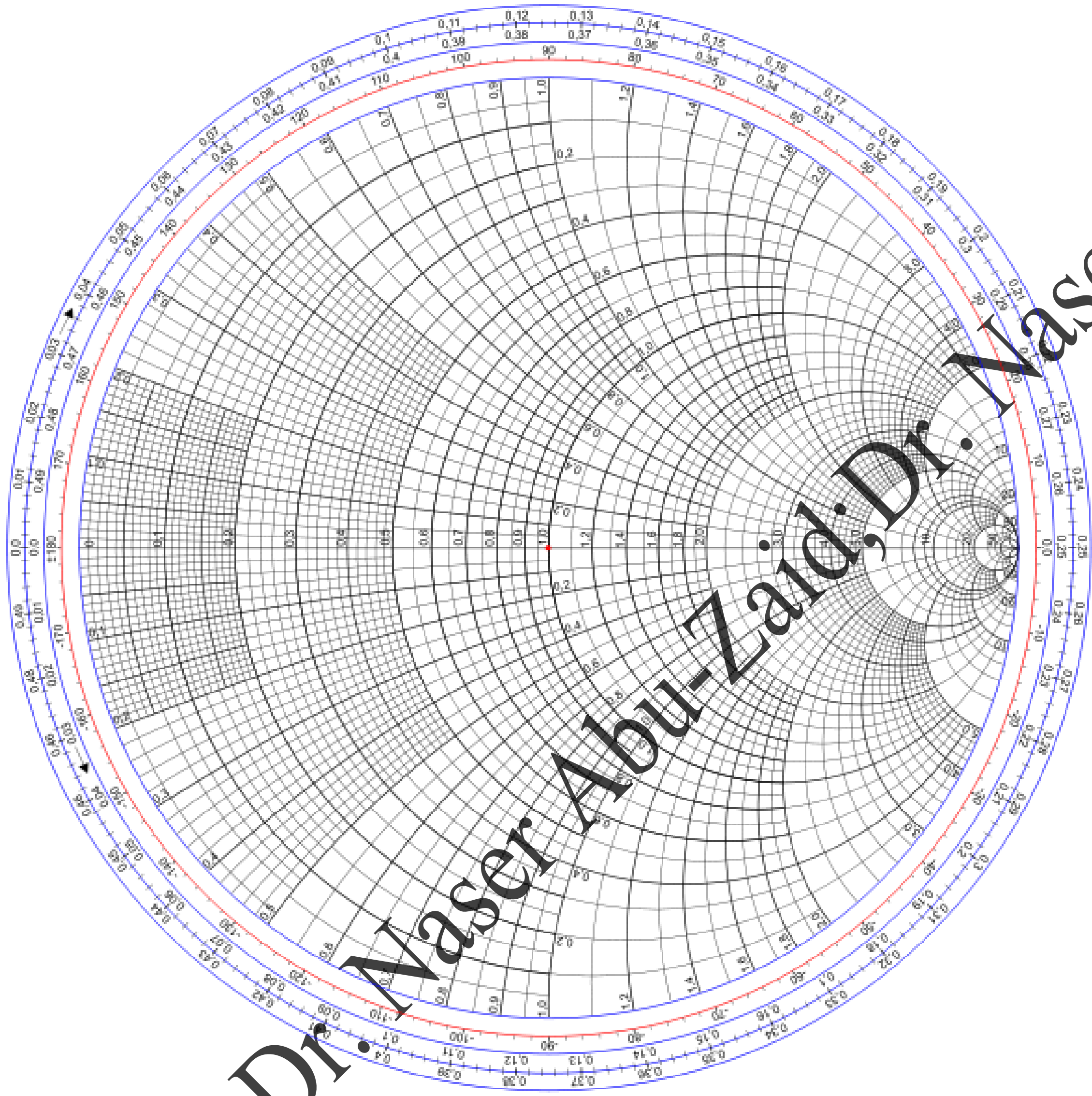
Center $= \left\{ 1 , \frac{1}{x} \right\}$ **Radius** $= \frac{1}{x}$

As the normalized reactance x varies from $-\infty$ to ∞, we obtain a family of arcs contained inside the domain of the reflection coefficient $|\Gamma| \leq 1$.



Basic Smith Chart techniques for loss-less transmission lines

- **Given $Z(d) \Rightarrow$ Find $\Gamma(d)$**
  Given $\Gamma(d) \Rightarrow$ Find $Z(d)$
- **Given Γ_L or $Z_L \Rightarrow$ Find $\Gamma(d)$ and $Z(d)$ @ a specified d.**
  Given $\Gamma(d)$ or $Z(d) \Rightarrow$ Find Γ_L and Z_L
- **Find $dmax$ and $dmin$ (maximum and minimum locations for the voltage standing wave pattern)**
- **Find the Voltage Standing Wave Ratio s (VSWR)**
- **Given $Z(d) \Rightarrow$ Find $Y(d)$**
  Given $Y(d) \Rightarrow$ Find $Z(d)$



Given $Z(d) \Rightarrow$ Find $\Gamma(d)$

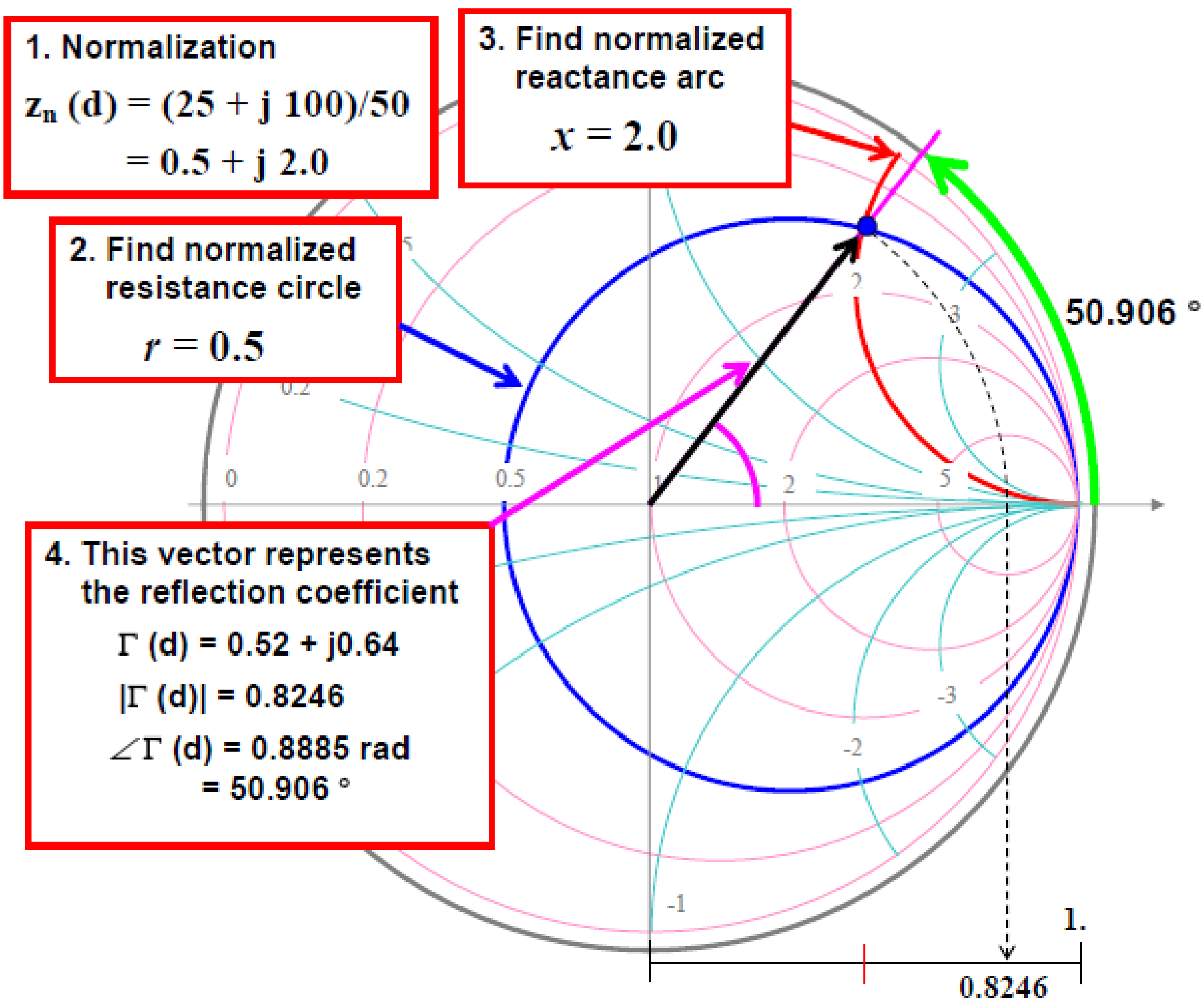
1. Normalize the impedance

$$z_n(\mathrm{d}) = \frac{Z(\mathrm{d})}{Z_0} = \frac{R}{Z_0} + j\,\frac{X}{Z_0} = r + j\,x$$

2. Find the circle of constant normalized resistance r
3. Find the arc of constant normalized reactance x
4. The intersection of the two curves indicates the reflection coefficient in the complex plane. The chart provides directly the magnitude and the phase angle of $\Gamma(d)$

Example: Find $\Gamma(d)$, given

$$Z(d) = 25 + j\,100\,\Omega \text{ with } Z_o = 50\,\Omega$$



Given $\Gamma(d) \Rightarrow$ Find $Z(d)$

1. Determine the complex point representing the given reflection coefficient $\Gamma(d)$ on the chart.
2. Read the values of the normalized resistance r and of the normalized reactance x that corresponds to the reflection coefficient point.
3. The normalized impedance is $z_n(d) = r + j\,x$ and the actual impedance is
$Z(d) = Z_o * z_n(d) = Z_o * (r + j\,x) = Z_o * r + j\,Z_o\,x$

Given Γ_L and/or $Z_L \Leftarrow\Rightarrow$ Find $\Gamma(d)$ and $Z(d)$

NOTE: the magnitude of the reflection coefficient is constant along a loss-less transmission line terminated by a specified load, since

$$|\Gamma(d)| = |\Gamma_L \, exp(-j2\beta d)| = |\Gamma_L|$$

Therefore, on the complex plane, a circle with center at the origin and radius $|\Gamma_L|$ represents all possible reflection coefficients found along the transmission line. When the circle of constant magnitude of the reflection coefficient is drawn on the Smith chart, one can determine the values of the line impedance at any location.

The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_L and the normalized load impedance Z_L on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_L|$.
3. Starting from the point representing the load, travel on the circle in the clockwise direction (wavelengths toward generator), by an angle

$$\theta = 2\,\beta\,d = 2\,\frac{2\pi}{\lambda}\,d$$

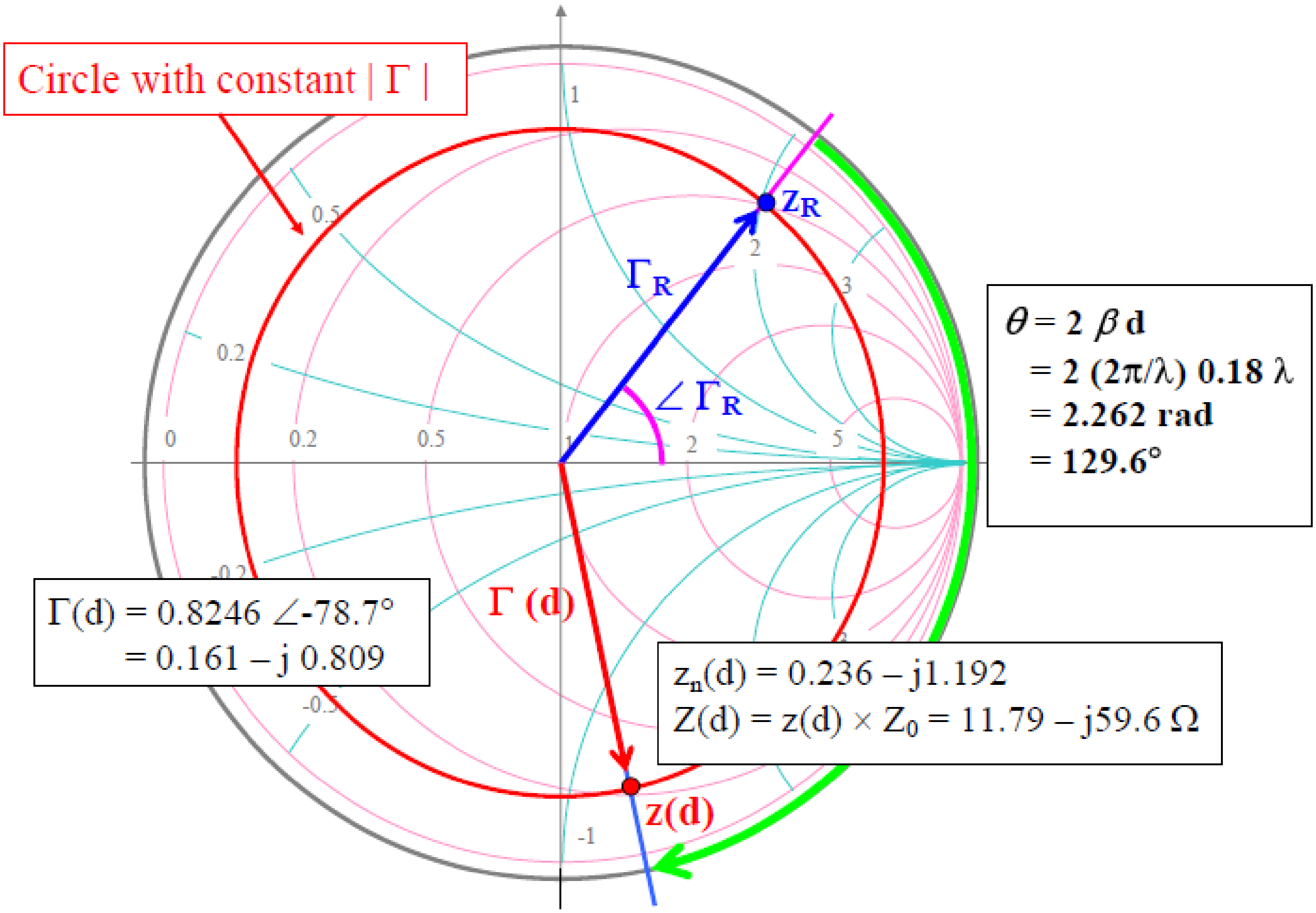
4. The new location on the chart corresponds to location d on the transmission line. Here, the values of $\Gamma(d)$ and $Z(d)$ can be read from the chart as before.

Example: Given

$$Z_L = 25 + j\,100\,\Omega \quad \text{with} \quad Z_o = 50\,\Omega$$

find

$$Z(d) \quad \text{and} \quad \Gamma(d) \quad \text{for} \quad d = 0.18\lambda$$

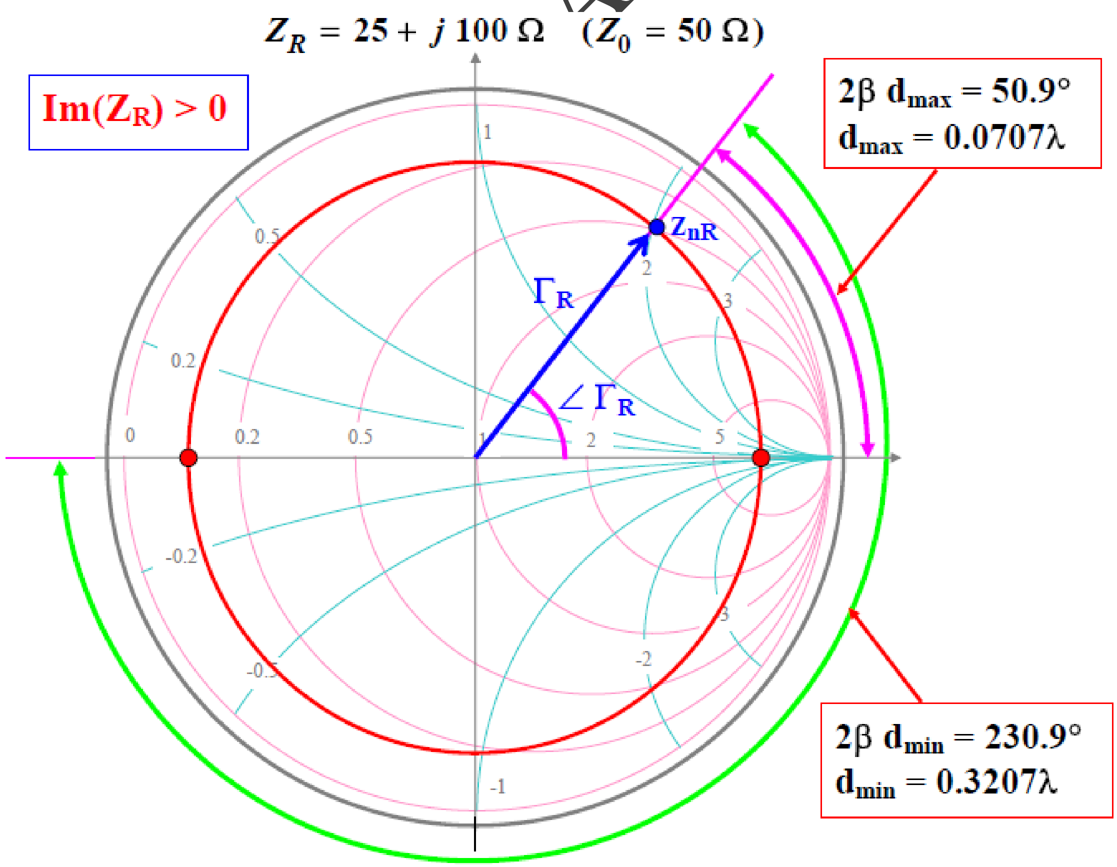


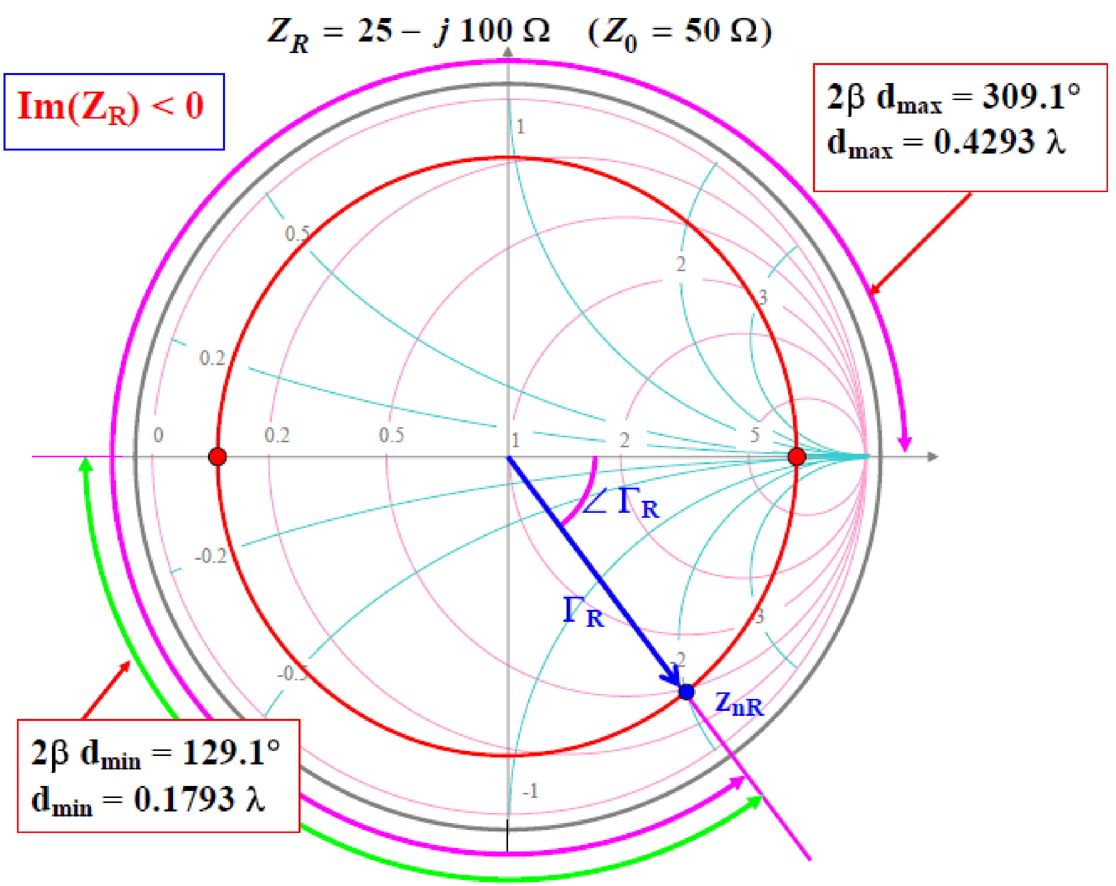
Given Γ_L and/or $Z_L \Rightarrow$ Find $dmax$ and $dmin$

1. Identify on the Smith chart the load reflection coefficient Γ_L or the normalized load impedance Z_L .
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_L|$. The circle intersects the real axis of the reflection coefficient at two points which identify $dmax$ (when $\Gamma(d) = Real\ positive$) and $dmin$ (when $\Gamma(d) = Real\ negative$)
3. A commercial Smith chart provides an outer graduation where the distances normalized to the wavelength can be read directly. The angles, between the vector Γ_L and the real axis, also provide a way to compute $dmax$ and $dmin$.

Example: Find $dmax$ and $dmin$ for inductive and capacitive loads

$$Z_L = 25 + j\,100\,\Omega\ ;$$
And $Z_L = 25 - j100\Omega$
Where ($Z0 = 50\,\Omega$)





Given Γ_L and Z_L ⇒ Find the Voltage Standing Wave Ratio s (VSWR)

The Voltage standing Wave Ratio or VSWR is defined as

$$VSWR = s = \frac{|V_s(z)|_{\max}}{|V_s(z)|_{\min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

The normalized impedance at a maximum location of the standing wave pattern is given by

$$z_n(d_{max}) = \frac{1+\Gamma(d_{max})}{1-\Gamma(d_{max})} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = VSWR \text{ !!!}$$

This quantity is always real and ≥ 1. ***The VSWR is simply obtained on the Smith chart, by reading the value of the (real) normalized impedance, at the location dmax*** where Γ is real and positive.

The graphical step-by-step procedure is:

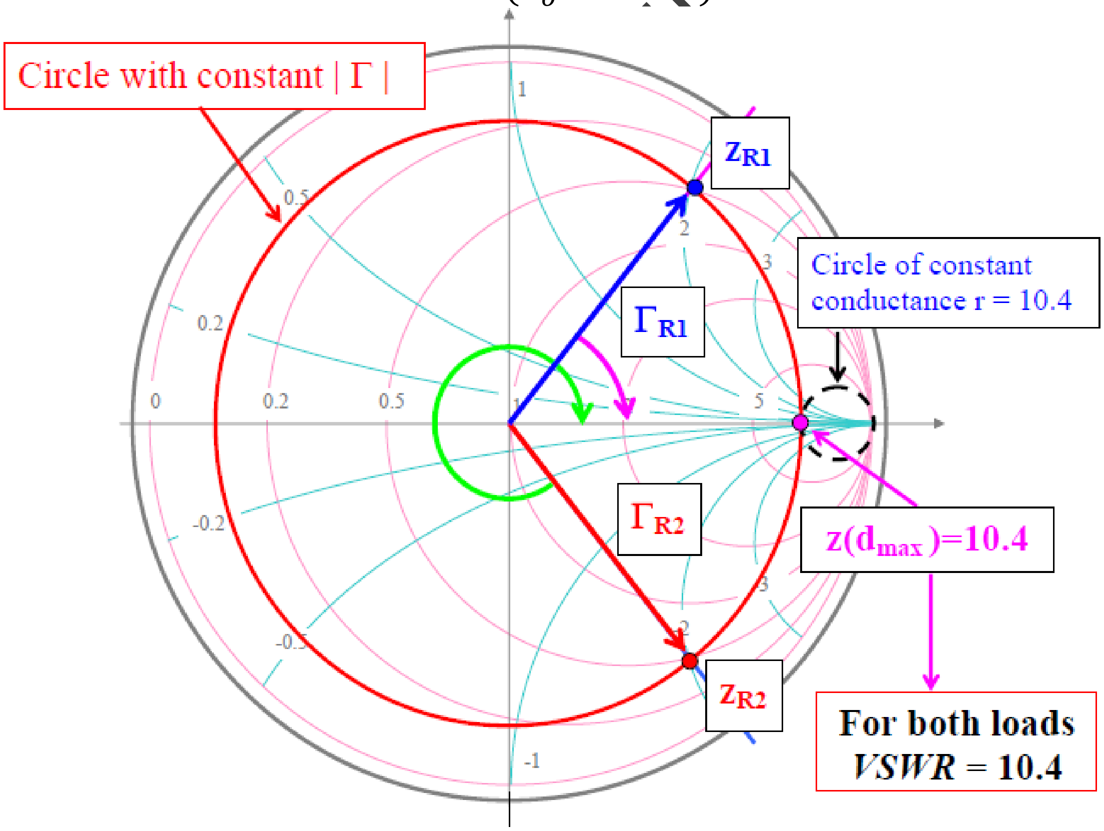
1. Identify the load reflection coefficient Γ_L and the normalized load impedance Z_L on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_L|$.
3. Find the intersection of this circle with the real positive axis for the reflection coefficient (corresponding to the transmission line location $dmax$).
4. A circle of constant normalized resistance will also intersect this point. Read or interpolate the value of the normalized resistance to determine the *VSWR*.

Example: Find the *VSWR* for two different loads

$$Z_{L1} = 25 + j\,100\,\Omega,$$

And $Z_{L2} = 25 - j100\Omega$

Where ($Z_o = 50\,\Omega$)



Given $\boldsymbol{Z(d)} \Leftarrow\Rightarrow$ Find $\boldsymbol{Y(d)}$

Review the impedance-admittance terminology:

Impedance = Resistance + j Reactance

$$Z = R + jX$$

Admittance = Conductance + j Susceptance

$$Y = G + jB$$

Note: The normalized impedance and admittance are defined as

$$z_n(d) = \frac{1+\Gamma(d)}{1-\Gamma(d)} \qquad y_n(d) = \frac{1-\Gamma(d)}{1+\Gamma(d)}$$

Keep in mind that the equality

$$z_n\left(d+\frac{\lambda}{4}\right) = y_n(d)$$

is only valid for normalized impedance and admittance. The actual values are given by

$$Z\left(d+\frac{\lambda}{4}\right) = Z_0 \cdot z_n\left(d+\frac{\lambda}{4}\right)$$

$$Y(d) = Y_0 \cdot y_n(d) = \frac{y_n(d)}{Z_0}$$

where $Y_o = \frac{1}{Z_o}$ is the characteristic admittance of the transmission line.

The graphical step-by-step procedure is:

1. Identify the load reflection coefficient Γ_L and the normalized load impedance Z_L on the Smith chart.
2. Draw the circle of constant reflection coefficient amplitude $|\Gamma(d)| = |\Gamma_L|$.
3. The normalized admittance is located at a point on the circle of constant $|\Gamma|$ which is diametrically opposite to the normalized impedance.

Example: Given
$Z_L = 25 + j\,100\,\Omega$ with $Z_o = 50\,\Omega$ find Y_L .