# UNIFORM PLANE WAVES (PROPAGATION IN FREE SPACE)

Starting with point form of Maxwell's equations for time varying fields in free space:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \bullet \mathbf{E} = 0$$

$$\nabla \bullet \mathbf{H} = 0$$

Let

$$\mathbf{E} = E_x(z)\hat{\mathbf{a}}_x$$

Then

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\hat{\mathbf{a}}_y = -\mu_o \frac{\partial \mathbf{H}}{\partial t} = -\mu_o \frac{\partial H_y(z)}{\partial t}\hat{\mathbf{a}}_y$$

And

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\hat{\mathbf{a}}_x = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \varepsilon_o \frac{\partial E_x}{\partial t}\hat{\mathbf{a}}_x$$

Collecting results

$$\frac{\partial E_x}{\partial z} = -\mu_o \frac{\partial H_y(z)}{\partial t} \qquad \frac{\partial H_y}{\partial z} = -\varepsilon_o \frac{\partial E_x}{\partial t}$$

***Good reminder of telegraphist equations!***

To obtain the wave equations, differentiate the first w.r.t z and the second w.r.t t and rearranging to get:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2} \begin{cases} \text{One dimensional} \\ \text{wave equation for } \mathbf{E} \end{cases}$$

Or reversing differentiations to get:

$$\frac{\partial^2 H_y}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 H_y}{\partial t^2} \begin{cases} \text{One dimensional} \\ \text{wave equation for } \mathbf{H} \end{cases}$$

And a general solution is given by:

$$E_x(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = E^+ + E^-$$

From which the velocity of wave propagation may be deduced (by substituting $f_1$ in the wave equation, performing the indicated diff's)

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3\times 10^8 \left( m/s \right) = c$$

**TEM** waves: **T**ransverse **E**lectro**M**agnetic waves implies **E** is perpendicular to **H** and both lying in a transverse plane (a plane normal to the direction of propagation)

**Uniform Plane Waves UPW**: **E** and **H** fields have constant magnitude and phase in the transverse plane. (Constant phase and amplitude).

For sinusoidal waves:

$$E_{x_{total}}(z,t) = E_x(z,t) + E_x'(z,t)$$

$$= |E_{xo}| \cos\left[\omega\left(t - \frac{z}{v_p}\right) + \varphi_1\right] + |E_{xo}'| \cos\left[\omega\left(t + \frac{z}{v_p}\right) + \varphi_2\right]$$

$$= |E_{xo}| \cos[\omega t - k_o z + \varphi_1] + |E_{xo}'| \cos[\omega t + k_o z + \varphi_2]$$

$$\omega \rightarrow \left( rad/s \right) \rightarrow \text{phase shift per unit time}$$

$$k_o \rightarrow \left( rad/m \right) \rightarrow \text{phase shift per unit distance}$$

constant phase implies

$$\omega t - k_o z + \varphi_1 = \text{constant}$$

$$\therefore \quad \frac{d}{dt}[\omega t - k_o z + \varphi_1] = \frac{d}{dt}[\text{constant}] = 0$$

$$\therefore \quad \frac{dz}{dt} = \frac{\omega}{k_o} = v_p = c (\text{in free space})$$

The *wave number in free space* is defined as:

$$k_o = \frac{\omega}{c} \left( rad/m \right)$$

*The **wavenumber** is a property of a wave, its **spatial frequency**, that is proportional to the reciprocal of the wavelength. It is also the magnitude of the **wave vector** (to be seen later). The wavenumber has dimensions of reciprocal length, so its SI unit is $m^{-1}$.*

***Simply the number of wavelengths per 2π units of distance.***

Also the *wave length* is given by:

$$\underbrace{\lambda_o}_{\substack{\text{distance}\\ \text{over which}\\ \text{the spatial}\\ \text{phase shifts}\\ \text{by } 2\pi}} = \frac{2\pi}{k_o} = \frac{c}{f}\,(m)$$

Maxwell's equations and the wave equations may be written in ***frequency domain*** with the help of the transformation;

$$\frac{\partial}{\partial t} \to j\omega$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \quad \to \quad \nabla \times \mathbf{E}_s = -j\omega\mu_o \mathbf{H}_s$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \quad \to \quad \nabla \times \mathbf{H}_s = j\omega\varepsilon_o \mathbf{E}_s$$

$$\nabla \bullet \mathbf{E}_s = 0$$

$$\nabla \bullet \mathbf{H}_s = 0$$

## 3D WAVE EQUATIONS (FREE SPACE)

Taking the curl of the first equation, namely

$$\nabla \times \nabla \times \mathbf{E}_s = -j\,\omega\,\mu_o \nabla \times \mathbf{H}_s$$

Using the identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \bullet \mathbf{F}) - \nabla^2 \mathbf{F}$$

Substituting

$$\nabla \times \nabla \times \mathbf{E}_s = \nabla(\nabla \bullet \mathbf{E}_s) - \nabla^2 \mathbf{E}_s = -j\,\omega\,\mu_o \nabla \times \mathbf{H}_s$$

Using the rest of Maxwell's equations

$$-\nabla^2 \mathbf{E}_s = -j\,\omega\,\mu_o \nabla \times \mathbf{H}_s$$

$$-\nabla^2 \mathbf{E}_s = -j\,\omega\,\mu_o (-j\omega\varepsilon_o \mathbf{E}_s)$$

$$\nabla^2 \mathbf{E}_s = -\omega^2 \mu_o \varepsilon_o \mathbf{E}_s$$

$$\underbrace{\nabla^2 \mathbf{E}_s = -k_o^2 \mathbf{E}_s}_{\textit{Vector Helmholtz equation}\ (\text{Wave eqation for } \mathbf{E})}$$

With

$$k_o = \frac{\omega}{v_p} = \omega\sqrt{\varepsilon_o \mu_o}\ \left(rad/m\right)$$

Similar approach may be followed to obtain the wave equation for the magnetic field

$$\underbrace{\nabla^2 \mathbf{H}_s = -k_o^2 \mathbf{H}_s}_{\text{Vector Helmholtz equation ( Wave eqation for } \mathbf{H})}$$

## RELATION BETWEEN *E* AND *H*

For the previous assumption $\mathbf{E} = E_x(z)\hat{\mathbf{a}}_x$, the wave equation reduces to:

$$\frac{d^2 E_{xs}(z)}{dz^2} = -k_o^2 E_{xs}(z)$$

With the frequency domain solution given by:

$$E_{xs}(z) = E_{xo}\, e^{-jk_o z} + E'_{x0}\, e^{jk_o z}$$

But from Maxwell's equation

$$\nabla \times \mathbf{E}_s = -j\,\omega\,\mu_o \mathbf{H}_s$$

$$\frac{dE_{xs}(z)}{dz} = -j\omega\mu_o H_{ys}(z)$$

Substituting the solution

$$\frac{d\left(E_{xo}\, e^{-jk_o z} + E'_{x0}\, e^{jk_o z}\right)}{dz} = -j\omega\mu_o H_{ys}(z)$$

Differentiating and solving for $H_{ys}(z)$

$$H_{ys}(z) = \sqrt{\frac{\varepsilon_o}{\mu_o}} E_{xo}\, e^{-jk_o z} - \sqrt{\frac{\varepsilon_o}{\mu_o}} E'_{x0}\, e^{jk_o z}$$

Identifying

$$H_{yf}(z) = H_{yo}\, e^{-jk_o z}$$

$$H_{yb}(z) = H'_{y0}\, e^{jk_o z}$$

$$H_{yo} = \sqrt{\frac{\varepsilon_o}{\mu_o}} E_{xo}$$

$$H'_{yo} = -\sqrt{\frac{\varepsilon_o}{\mu_o}} E'_{x0}$$

And the *intrinsic impedance of free space* is

$$\eta_o = \frac{E_{xo}}{H_{yo}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi = 377\,(\Omega)$$

Or

$$\eta_o = -\frac{E'_{xo}}{H'_{yo}} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\,(\Omega)$$

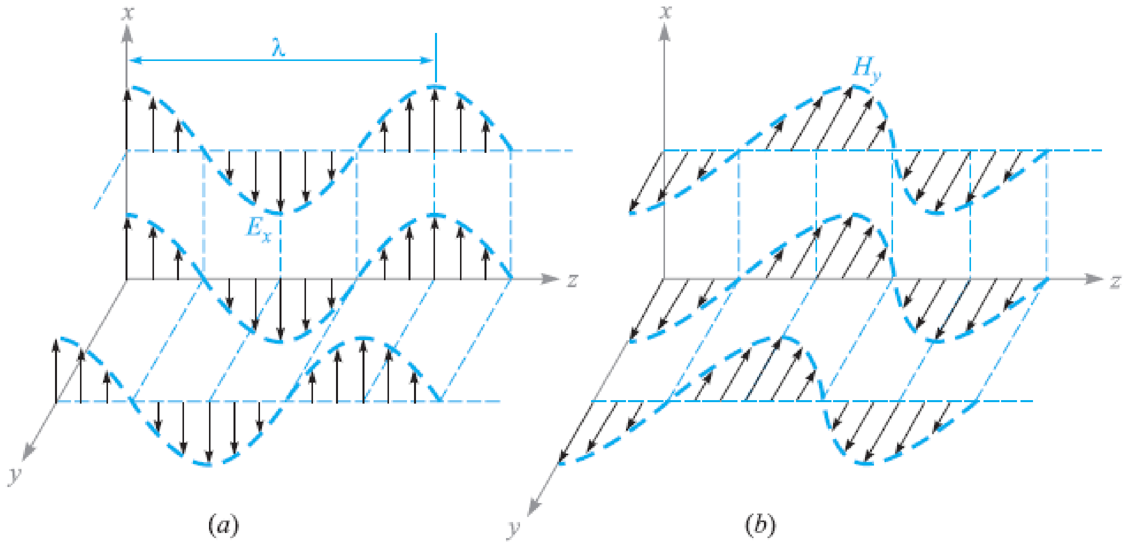



**Figure 11.1** (a) Arrows represent the instantaneous values of $E_{x0}\cos[\omega(t-z/c)]$ at $t=0$ along the $z$ axis, along an arbitrary line in the $x=0$ plane parallel to the $z$ axis, and along an arbitrary line in the $y=0$ plane parallel to the $z$ axis. (b) Corresponding values of $H_y$ are indicated. Note that $E_x$ and $H_y$ are in phase at any point in time.

*It can be shown that:*

$$\mathbf{H} = \frac{\hat{\mathbf{a}}_n \times \mathbf{E}}{\eta_o} \quad or \quad \mathbf{E} = -\eta_o\,\hat{\mathbf{a}}_n \times \mathbf{H}$$

Or

$$\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_E \times \hat{\mathbf{a}}_H$$

Where

$\hat{\mathbf{a}}_n$ : unit vector in the direction of propagation.

$\hat{\mathbf{a}}_E$ : unit vector in the direction of ***E***.

$\hat{\mathbf{a}}_H$ : unit vector in the direction of ***H***.

**EXAMPLE 11.1**

Let us express $\mathcal{E}_y(z,t) = 100\cos(10^8 t - 0.5z + 30^\circ)$ V/m as a phasor.

**Solution.** We first go to exponential notation,

$$\mathcal{E}_y(z,t) = \text{Re}\left[100e^{j(10^8 t - 0.5z + 30^\circ)}\right]$$

and then drop Re and suppress $e^{j10^8 t}$, obtaining the phasor

$$E_{ys}(z) = 100e^{-j0.5z + j30^\circ}$$

**EXAMPLE 11.2**

Given the complex amplitude of the electric field of a uniform plane wave, $\mathbf{E}_0 = 100\mathbf{a}_x + 20\angle 30^\circ \mathbf{a}_y$ V/m, construct the phasor and real instantaneous fields if the wave is known to propagate in the forward $z$ direction in free space and has frequency of 10 MHz.

**Solution.** We begin by constructing the general phasor expression:

$$\mathbf{E}_s(z) = \left[100\mathbf{a}_x + 20e^{j30^\circ}\mathbf{a}_y\right]e^{-jk_0 z}$$

where $k_0 = \omega/c = 2\pi \times 10^7/3 \times 10^8 = 0.21$ rad/m. The real instantaneous form is then found through the rule expressed in Eq. (19):

$$\begin{aligned}\mathcal{E}(z,t) &= \text{Re}\left[100e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_x + 20e^{j30^\circ}e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_y\right]\\ &= \text{Re}\left[100e^{j(2\pi\times10^7 t - 0.21z)}\mathbf{a}_x + 20e^{j(2\pi\times10^7 t - 0.21z + 30^\circ)}\mathbf{a}_y\right]\\ &= 100\cos(2\pi\times10^7 t - 0.21z)\mathbf{a}_x + 20\cos(2\pi\times10^7 t - 0.21z + 30^\circ)\mathbf{a}_y\end{aligned}$$

**Ex. 12.1:** Let $\mathbf{H}_s = \left(2\angle -40^o\,\hat{\mathbf{a}}_x - 3\angle 20^o\,\hat{\mathbf{a}}_y\right)e^{-j0.07z}\left(A/m\right)$ for a uniform plane wave traveling in free space. Find:

1) $\omega$
2) $H_x$ @ $P(1,2,3, t = 31ns)$
3) $|\mathbf{H}|$ @ $t = 0$ @ the origin

## PROPAGATION IN DIELECTRICS

Assuming a *simple dielectric*, the wave equation is written as:

$$\nabla^2\mathbf{E}_s = -k^2\,\mathbf{E}_s \qquad (\text{Wave eqation for } \mathbf{E})$$

Where $k$ is the *wave number*.

$$k = \omega\sqrt{\varepsilon\mu}$$

Allowing the *permittivity* to be a *complex constant (to be explained later)*, implies that the wave number may be complex and it is called the ***complex propagation constant***.

*The **propagation constant** of an electromagnetic wave is a measure of the change undergone by the amplitude of the wave as it propagates in a given direction. The propagation constant itself **measures change per metre** but is otherwise dimensionless. The quantity measured, such as voltage or electric field intensity, is expressed as a sinusoidal phasor. The phase of the sinusoid varies with distance which results in the propagation constant being a complex number, the imaginary part being caused by the phase change.*

For a One dimensional problem $\mathbf{E}_s = E_x(z)\hat{\mathbf{a}}_x$, the wave equation reduces to

$$\frac{d^2E_{xs}(z)}{dz^2} = -k^2E_{xs}(z)$$

Define

$$\gamma = jk = \alpha + j\beta$$

So, the solution is given by

$$\begin{aligned} E_{xs}(z) &= E_{xo}\, e^{-\gamma z} + E'_{xo}\, e^{\gamma z} \\ &= E_{xo}\, e^{-jk z} + E'_{xo}\, e^{jk z} \\ &= E_{xo}\, e^{-\alpha z} e^{-j\beta z} + E'_{xo} e^{\alpha z}\, e^{j\beta z} \end{aligned}$$

Transferring to time domain, and considering only the forward part:

$$E_x(z,t) = E_{xo} e^{-\alpha z}\cos(\omega t - \beta z)$$

Define the *complex permittivity* (dipole oscillations and conduction electrons and holes) as:

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_o\varepsilon'_r - j\varepsilon_o\varepsilon''_r = \varepsilon_o\left(\varepsilon'_r - j\varepsilon''_r\right)$$

$$k = \omega\sqrt{\varepsilon\mu} = \omega\sqrt{(\varepsilon' - j\varepsilon'')\mu} = \omega\sqrt{\mu\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)}$$

$$= \omega\sqrt{\mu\varepsilon'}\left[\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)\right]^{\frac{1}{2}}$$

With

$$\alpha = \text{Re}\{\gamma\} = \text{Re}\{jk\} = \omega\sqrt{\frac{\varepsilon'\mu}{2}}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]^{\frac{1}{2}}$$

$$\beta = \text{Im}\{\gamma\} = \text{Im}\{jk\} = \omega\sqrt{\frac{\varepsilon'\mu}{2}}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]^{\frac{1}{2}}$$

Clearly from the time domain expression of $E_x(z,t)$, the *phase velocity* is given by:

$$v_p = \frac{\omega}{\beta}\left(m/s\right)$$

And the *wave length* is (distance required to change the phase by $2\pi$):

$$\lambda = \frac{2\pi}{\beta}(m)$$

And the magnetic field associated with the forward propagating part is: (can be found through the use of Maxwell's equations)

$$H_{ys}(z) = \frac{E_{xo}}{\eta}e^{-\gamma z} = \frac{E_{xo}}{\eta}e^{-jk z}$$

$$= \frac{E_{xo}}{\eta}e^{-\alpha z}e^{-j\beta z} = \frac{E_{xo}}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\theta_\eta}$$

With the *intrinsic impedance* being a complex quantity, given by:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\frac{1}{\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}}} = |\eta|e^{j\theta_\eta}$$

Since

$$E_x(z,t) = E_{xo} e^{-\alpha z} \cos(\omega t - \beta z)$$

$$H_y(z,t) = \frac{E_{xo}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)$$

Then $E_x$ leads $H_y$ by $\theta_\eta$. And you may do the same for the backward wave.

## Lossless medium (Perfect dielectric)

$$\varepsilon'' = 0 \rightarrow \varepsilon = \varepsilon' = \varepsilon_o \varepsilon_r$$

$$\alpha = \mathrm{Re}\{\gamma\} = \mathrm{Re}\{jk\} = \omega \sqrt{\frac{\varepsilon' \mu}{2}} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right]^{\frac{1}{2}} = 0$$

$$\beta = \mathrm{Im}\{\gamma\} = \mathrm{Im}\{jk\} = \omega \sqrt{\frac{\varepsilon' \mu}{2}} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right]^{\frac{1}{2}} = \omega \sqrt{\varepsilon' \mu}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon' \mu}} = \frac{c}{\sqrt{\varepsilon_r' \mu_r}} \left( m/s \right)$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega \sqrt{\varepsilon' \mu}} = \frac{c}{f \sqrt{\varepsilon_r' \mu_r}} = \frac{\lambda_o}{\sqrt{\varepsilon_r' \mu_r}} (m)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}} \frac{1}{\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}}} = \sqrt{\frac{\mu}{\varepsilon'}} = \sqrt{\frac{\mu_o \mu_r}{\varepsilon_o \varepsilon_r'}}$$

$$= \eta_o \sqrt{\frac{\mu_r}{\varepsilon_r'}} = |\eta| e^{j\theta_\eta}$$

$$\rightarrow |\eta| = \eta_o \sqrt{\frac{\mu_r}{\varepsilon_r'}} \quad \& \quad \theta_\eta = 0$$

$$\eta = \eta_o \sqrt{\frac{\mu_r}{\varepsilon_r^{'}}} \quad \& \quad \theta_\eta = 0$$

$E_x$ and $H_y$ are in phase



**EXAMPLE 11.3**

Let us apply these results to a 1-MHz plane wave propagating in fresh water. At this frequency, losses in water are negligible, which means that we can assume that $\epsilon'' \doteq 0$. In water, $\mu_r = 1$ and at 1 MHz, $\epsilon_r' = 81$.

**Solution.** We begin by calculating the phase constant. Using (45) with $\epsilon'' = 0$, we have

$$\beta = \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r'} = \frac{\omega\sqrt{\epsilon_r'}}{c} = \frac{2\pi \times 10^6\sqrt{81}}{3.0 \times 10^8} = 0.19 \text{ rad/m}$$

Using this result, we can determine the wavelength and phase velocity:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{.19} = 33 \text{ m}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{.19} = 3.3 \times 10^7 \text{ m/s}$$

The wavelength in air would have been 300 m. Continuing our calculations, we find the intrinsic impedance using (48) with $\epsilon'' = 0$:

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \frac{\eta_0}{\sqrt{\epsilon_r'}} = \frac{377}{9} = 42\ \Omega$$

If we let the electric field intensity have a maximum amplitude of 0.1 V/m, then

$$E_x = 0.1\cos(2\pi 10^6 t - .19z) \text{ V/m}$$

$$H_y = \frac{E_x}{\eta} = (2.4 \times 10^{-3})\cos(2\pi 10^6 t - .19z) \text{ A/m}$$

**D11.3.** A 9.375-GHz uniform plane wave is propagating in polyethylene (see Appendix C). If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find: (*a*) the phase constant; (*b*) the wavelength in the polyethylene; (*c*) the velocity of propagation; (*d*) the intrinsic impedance; (*e*) the amplitude of the magnetic field intensity.

**Ans.** 295 rad/m; 2.13 cm; $1.99 \times 10^8$ m/s; 251 Ω; 1.99 A/m

**EXAMPLE 11.4**

We again consider plane wave propagation in water, but at the much higher microwave frequency of 2.5 GHz. At frequencies in this range and higher, dipole relaxation and resonance phenomena in the water molecules become important.[2] Real and imaginary parts of the permittivity are present, and both vary with frequency. At frequencies below that of visible light, the two mechanisms together produce a value of $\epsilon''$ that increases with increasing frequency, reaching a maximum in the vicinity of $10^{13}$ Hz. $\epsilon'$ decreases with increasing frequency, reaching a minimum also in the vicinity of $10^{13}$ Hz. Reference 3 provides specific details. At 2.5 GHz, dipole relaxation effects dominate. The permittivity values are $\epsilon'_r = 78$ and $\epsilon''_r = 7$. From (44), we have

$$\alpha = \frac{(2\pi \times 2.5 \times 10^9)\sqrt{78}}{(3.0 \times 10^8)\sqrt{2}} \left( \sqrt{1 + \left(\frac{7}{78}\right)^2} - 1 \right)^{1/2} = 21 \text{ Np/m}$$

This first calculation demonstrates the operating principle of the *microwave oven*. Almost all foods contain water, and so they can be cooked when incident microwave radiation is absorbed and converted into heat. Note that the field will attenuate to a value of $e^{-1}$ times its initial value at a distance of $1/\alpha = 4.8$ cm. This distance is called the *penetration depth* of the material, and of course it is frequency-dependent. The 4.8 cm depth is reasonable for cooking food, since it would lead to a temperature rise that is fairly uniform throughout the depth of the material. At much higher frequencies, where $\epsilon''$ is larger, the penetration depth decreases, and too much power is absorbed at the surface; at lower frequencies, the penetration depth increases, and not enough overall absorption occurs. Commercial microwave ovens operate at frequencies in the vicinity of 2.5 GHz.

Using (45), in a calculation very similar to that for $\alpha$, we find $\beta = 464$ rad/m. The wavelength is $\lambda = 2\pi/\beta = 1.4$ cm, whereas in free space this would have been $\lambda_0 = c/f = 12$ cm.

Using (48), the intrinsic impedance is found to be

$$\eta = \frac{377}{\sqrt{78}} \frac{1}{\sqrt{1 - j(7/78)}} = 43 + j1.9 = 43\angle 2.6^\circ \ \Omega$$

and $E_x$ leads $H_y$ in time by 2.6° at every point.



## (Lossy Dielectrics)

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon \mathbf{E}_s = j\omega(\varepsilon' - j\varepsilon'')\mathbf{E}_s$$
$$= j\omega\varepsilon' \mathbf{E}_s + \omega\varepsilon'' \mathbf{E}_s$$

But on the other hand,

$$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega\varepsilon \mathbf{E}_s = \sigma \mathbf{E}_s + j\omega\varepsilon' \mathbf{E}_s$$

Comparing the two equations

$$\omega\varepsilon'' = \sigma \rightarrow \varepsilon'' = \frac{\sigma}{\omega}$$

Also from the second equation

$$\frac{J_c}{J_d} = \frac{\varepsilon''}{j\varepsilon'} = \frac{\sigma}{j\omega\varepsilon'}$$

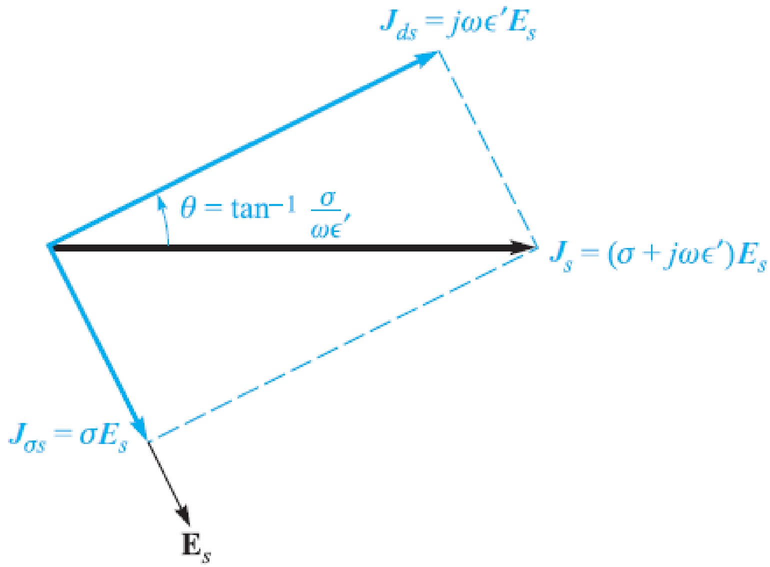



**Figure 11.2** The time-phase relationship between $\mathbf{J}_{ds}$, $\mathbf{J}_{\sigma s}$, $\mathbf{J}_s$, and $\mathbf{E}_s$. The tangent of $\theta$ is equal to $\sigma/\omega\epsilon'$, and $90° - \theta$ is the common power-factor angle, or the angle by which $\mathbf{J}_s$ leads $\mathbf{E}_s$.

$J_c$ and $J_d$ are $90^o$ out of time phase, and we identify the material as having *large losses or small losses depending on the magnitude of the loss tangent* defined by:

$$\text{Loss tangnet} = \tan(\theta) = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon'}$$

$\theta$ is the angle by which $\mathbf{J}_d$ lead the total current density $\mathbf{J}$

## *Good Dielectric Approximations*

$$\frac{\varepsilon''}{\varepsilon'} << 1 \quad or \quad \frac{\sigma}{\omega\varepsilon'} << 1 \quad \text{Small losses}$$

$$\gamma = jk = j\omega\sqrt{\varepsilon\mu} = j\omega\sqrt{\mu\varepsilon'}\left[\left(1 - j\frac{\sigma}{\omega\varepsilon'}\right)\right]^{\frac{1}{2}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon'}}\left[1 - j\frac{\sigma}{\omega\varepsilon'}\right]^{-\frac{1}{2}}$$

The above two *exact expressions* may be *approximated* using the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots, |x| << 1$$

Hence

$$\alpha = \mathrm{Re}\{\gamma\} = \mathrm{Re}\{jk\} \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon'}}$$

$$\beta = \mathrm{Im}\{\gamma\} = \mathrm{Im}\{jk\} \approx \omega\sqrt{\varepsilon'\mu}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon'}\right)^2\right]$$

$$\approx \omega\sqrt{\varepsilon'\mu}$$

$$\eta \approx \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\sigma}{2\omega\varepsilon'}\right)$$

*Why Complex Permittivity?*

*Loss mechanism occurs in dielectrics even in the absence of free electrons $(\sigma = 0)$, this is due to rotation of the dipoles to align with applied time varying field or due to the net shift of the electron cloud with respect to the positive nucleus. At high frequencies the polarization $(\mathbf{P})$ of the material is out of time phase with applied field. This loss mechanism is modeled by a complex permittivity, as shown previously, even with zero conductivity*

$$\varepsilon = \varepsilon' - j\varepsilon''$$

So, again from Amper's law

$$\nabla \times \mathbf{H}_s = j\omega(\varepsilon' - j\varepsilon'')\mathbf{E}_s = \underbrace{j\omega\varepsilon'\mathbf{E}_s}_{\substack{\text{Energy}\\ \text{storage}\\ \text{mechanism}\\ \mathbf{J}_d}} + \underbrace{\omega\varepsilon''\mathbf{E}_s}_{\substack{\text{Loss mechanism}\\ \text{or, may be}\\ \text{treated}\\ \text{as dielectric}\\ \text{conductivity}}}$$

And the loss tangent shown earlier is:

$$\text{Loss tangnet} = \tan(\theta) = \frac{\varepsilon''}{\varepsilon'}$$

*Even if the conductivity is nonzero $(\sigma \neq 0)$ but quite small, still we may write Amper's law as:*

$$\nabla \times \mathbf{H}_s = \sigma \mathbf{E}_s + j\omega(\varepsilon' - j\varepsilon'')\mathbf{E}_s$$

$$= \underbrace{\sigma \mathbf{E}_s}_{\substack{\text{Conductor loss mechanism} \\ \mathbf{J}_c}} + \underbrace{j\omega\varepsilon'\mathbf{E}_s}_{\substack{\text{Energy storage mechanism} \\ \mathbf{J}_d}} + \underbrace{\omega\varepsilon''\mathbf{E}_s}_{\substack{\text{Dielectric Loss mechanism or, maybe treated as} \\ (\textbf{dielectric conductivity})}}$$

So, we may write

$$\nabla \times \mathbf{H}_s = \sigma \mathbf{E}_s + j\omega(\varepsilon' - j\varepsilon'')\mathbf{E}_s$$

$$= \underbrace{(\sigma + \omega\varepsilon'')\mathbf{E}_s}_{\substack{\text{Conductor and dielectric loss mechanisms} \\ (\textbf{Effective conductivity})}} + \underbrace{j\omega \underbrace{\varepsilon'}_{\text{Effective permittivity}} \mathbf{E}_s}_{\substack{\text{Energy storage mechanism} \\ \mathbf{J}_d}}$$

And the loss tangent is defined as:

$$\text{Loss tangnet} = \tan(\theta) = \frac{\sigma + \omega\varepsilon''}{\omega\varepsilon'}$$

## Depth of penetration (Skin Depth)

Consider a forward traveling wave in a lossy dielectric;

$$E_x(z,t) = E_{xo} e^{-\alpha z} \cos(\omega t - \beta z)$$

At $z = \frac{1}{\alpha}$, the wave is attenuated by a factor of:

$$e^{-\alpha z} = e^{-\alpha\left(\frac{1}{\alpha}\right)} = e^{-1} = \underbrace{0.37}_{\text{Of it's maximum amplitude}}$$

The quantity

$$\delta = \frac{1}{\alpha}$$

is called the *skin depth, or depth of penetration*.

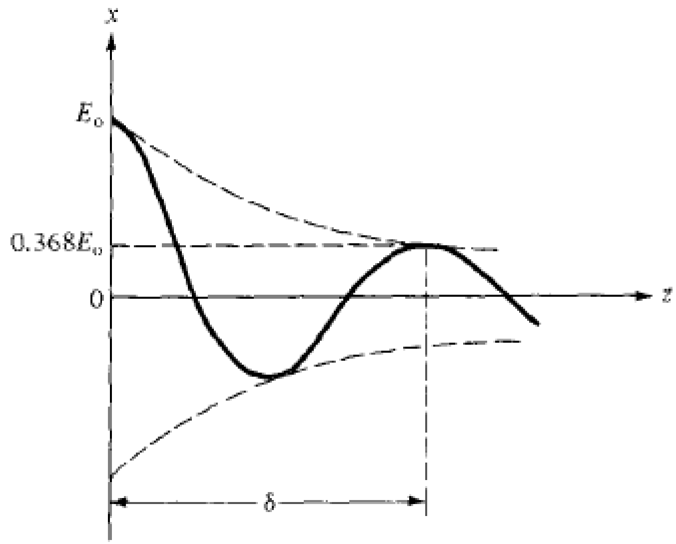



*Illustration of skin depth*



**EXAMPLE 11.5**

As a comparison, we repeat the computations of Example 11.4, using the approximation formulas (60*a*), (61), and (62*b*).

**Solution.** First, the loss tangent in this case is $\epsilon''/\epsilon' = 7/78 = 0.09$. Using (60), with $\epsilon'' = \sigma/\omega$, we have

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{1}{2}(7 \times 8.85 \times 10^{12})(2\pi \times 2.5 \times 10^{9})\frac{377}{\sqrt{78}} = 21 \text{ cm}^{-1}$$

We then have, using (61*b*),

$$\beta \doteq (2\pi \times 2.5 \times 10^{9})\sqrt{78}/(3 \times 10^{8}) = 464 \text{ rad/m}$$

Finally, with (62*b*),

$$\eta \doteq \frac{377}{\sqrt{78}}\left(1 + j\frac{7}{2 \times 78}\right) = 43 + j1.9$$

These results are identical (within the accuracy limitations as determined by the given numbers) to those of Example 11.4. Small deviations will be found, as the reader can verify by repeating the calculations of both examples and expressing the results to four or five significant figures. As we know, this latter practice would not be meaningful because the given parameters were not specified with such accuracy. Such is often the case, since measured values are not always known with high precision. Depending on how precise these values are, one can sometimes use a more relaxed judgment on when the approximation formulas can be used by allowing loss tangent values that can be larger than 0.1 (but still less than 1).

## POYNTING'S THEOREM (POWER THEOREM)

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Left dot both sides with $\mathbf{E}$, then using the identity

$$\mathbf{E} \bullet \nabla \times \mathbf{H} = \mathbf{H} \bullet \nabla \times \mathbf{E} - \nabla \bullet (\mathbf{E} \times \mathbf{H})$$

And with some vector manipulations, one can obtain (Follow text book)

$$\underbrace{-\nabla \bullet (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \bullet \mathbf{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{D} \bullet \mathbf{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B} \bullet \mathbf{H}\right)}_{\text{Differential form of the Poynting's Theorem}}$$

Integrating over a volume $v$ enclosed by a surface $s$:

$$-\iiint_v \nabla \bullet (\mathbf{E} \times \mathbf{H}) dv = \iiint_v \mathbf{E} \bullet \mathbf{J} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{D} \bullet \mathbf{E} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{B} \bullet \mathbf{H}$$

Upon using the divergence theorem for LHS

$$-\oiint_s (\mathbf{E} \times \mathbf{H}) \bullet ds = \iiint_v \mathbf{E} \bullet \mathbf{J} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{D} \bullet \mathbf{E} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{B} \bullet \mathbf{H} dv$$

$$\text{Total power flowing into volume} = \iiint_v \mathbf{E} \bullet \mathbf{J} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{D} \bullet \mathbf{E} dv + \frac{\partial}{\partial t}\iiint_v \frac{1}{2}\mathbf{B} \bullet \mathbf{H} dv$$
$$= -\oiint_s (\mathbf{E} \times \mathbf{H}) \bullet ds$$

$$\text{Total power flowing out of volume} = \oiint_s (\mathbf{E} \times \mathbf{H}) \bullet d\mathbf{s}$$

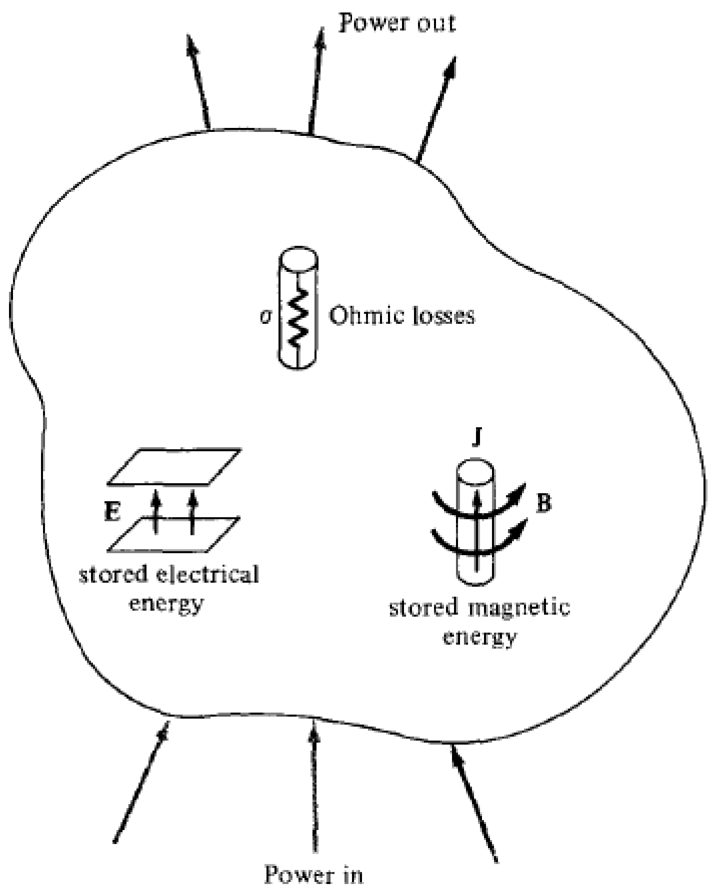



*Illustration of power balance for EM fields*

And the ***instantaneous Poynting vector*** **S** (or instantaneous power density) is defined as:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \left( \frac{W}{m^2} \right)$$

For UPW with

$$\mathbf{E}(z,t) = E_x(z,t)\hat{\mathbf{a}}_x$$
$$\mathbf{H}(z,t) = H_y(z,t)\hat{\mathbf{a}}_y$$

Then

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_x(z,t)\hat{\mathbf{a}}_x \times H_y(z,t)\hat{\mathbf{a}}_y = S_z\hat{\mathbf{a}}_z$$

1) Perfect Dielectric (forward part only):

$$E_x(z,t) = E_{xo}\cos(\omega t - \beta z)$$

$$H_y(z,t) = \frac{E_{xo}}{\eta}\cos(\omega t - \beta z)$$

$$S_z(z,t) = \frac{E_{xo}^2}{\eta}\cos^2(\omega t - \beta z)$$

**2)** Lossy Dielectric:

$$E_x(z,t) = E_{xo}e^{-\alpha z}\cos(\omega t - \beta z)$$

$$H_y(z,t) = \frac{E_{xo}}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$$

$$S_z(z,t) = \frac{E_{xo}^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

The ***average power density (time averaged Poynting's vector)*** is (for time harmonic case):

$$\langle \mathbf{S} \rangle = \frac{1}{T}\int_T \mathbf{S}dt =$$

$$\frac{1}{T}\int_T \frac{1}{2}\frac{E_{xo}^2}{|\eta|}e^{-2\alpha z}\left[\cos(2\omega t - 2\beta z - \theta_\eta) + \cos(\theta_\eta)\right]dt$$

$$\langle \mathbf{S} \rangle = \frac{1}{T}\int_T \mathbf{S}dt = \frac{1}{2}\frac{E_{xo}^2}{|\eta|}e^{-2\alpha z}\cos(\theta_\eta)$$

The above expression is easily evaluated using phasors by defining

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\}$$

Doing it for lossy dielectric (Sinusoidal wave)

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left\{ \left(E_{xo}e^{-\alpha z}e^{-j\beta z}\right) \times \left(\frac{E_{xo}}{|\eta|}e^{-\alpha z}e^{-j\beta z}e^{-j\theta_\eta}\right)^* \right\}$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}\frac{E_{xo}^2}{|\eta|}e^{-2\alpha z}\cos(\theta_\eta)$$

**Example 12.5:** At frequencies of 1, 100, and 3000MHz, the dielectric constant of ice made from pure water has values 0f 4.15, 3.45, and 3.2, respectively, while the loss tangent is 0.12, 0.035, and 0.0009, also respectively. If a UPW with amplitude of 100(V/m) @ z=0 is propagating through the ice, fine the time average power density @ z=0 and z=10m for each frequency.

Solution:

| $f$ $(MHz)$ | $\varepsilon_r^{'}$ | $\frac{\varepsilon''}{\varepsilon'}$ | $\langle \mathbf{S} \rangle$ @z=0 | $\langle \mathbf{S} \rangle$ @z=10m |
|---|---|---|---|---|
| 1 | 4.15 | 0.12 | 27.17 | 25.82 |
| 100 | 3.45 | 0.035 | | |
| 3000 | 3.2 | 0.0009 | 23.8 | |

Try to fill the rest.

## Good conductors Approximations (Skin effect)

$$\frac{\varepsilon''}{\varepsilon'} >> 1 \quad or \quad \frac{\sigma}{\omega\varepsilon'} >> 1 \quad \text{High losses}$$

$$\gamma = jk = j\omega\sqrt{\varepsilon\mu} = j\omega\sqrt{\mu\varepsilon'}\left[\left(1 - j\frac{\sigma}{\omega\varepsilon'}\right)\right]^{\frac{1}{2}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon'}}\left[1 - j\frac{\sigma}{\omega\varepsilon'}\right]^{-\frac{1}{2}}$$

The above two exact expressions may be approximated using the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, |x| << 1$$

Hence

$$\alpha = \beta \approx \sqrt{\pi f \mu\sigma}$$

$$\eta \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^o$$

Consider a forward traveling wave;

$$E_x(z,t) = E_{xo} e^{-\alpha z} \cos(\omega t - \beta z)$$

$$E_x(z,t) = E_{xo} e^{-z\sqrt{\pi f \mu \sigma}} \cos\left(\omega t - z\sqrt{\pi f \mu \sigma}\right)$$

$$J_x(z,t) = \sigma E_x = \sigma E_{xo} e^{-z\sqrt{\pi f \mu \sigma}} \cos\left(\omega t - z\sqrt{\pi f \mu \sigma}\right)$$

At $z = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} = \delta$, the wave is attenuated by a factor of;

$$e^{-\alpha z} = e^{-\alpha\left(\frac{1}{\alpha}\right)} = e^{-1} = \underbrace{0.37}_{\text{Of it's maximum amplitude}}$$

$$\delta = \frac{1}{\alpha} = \sqrt{\pi f \mu \sigma}$$

Which is the skin depth again, or depth of penetration.

For copper $\sigma = 5.8 \times 10^7 \left(S/m\right)$, $\mu = \mu_o$

$$\delta_{copper} = \frac{0.066}{\sqrt{f}}$$

| $f$ | $\delta_{copper}$ |
|---|---|
| $60Hz$ | $8.53mm$ |
| $10GHz$ | $6.6 \times 10^{-4} mm$ |

After a few skin depths within the conductor, *all fields are almost zero*.

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta$$

$$v_p = \frac{\omega}{\beta} = \omega\delta$$

$$\eta \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{1}{\sigma\delta}[1+j1] = \frac{\sqrt{2}\angle 45^o}{\sigma\delta}$$

If

$$E_x(z,t) = E_{xo}e^{-z\sqrt{\pi f\mu\sigma}}\cos\left(\omega t - z\sqrt{\pi f\mu\sigma}\right)$$

Then

$$H_y(z,t) = \frac{\sigma\delta}{\sqrt{2}}E_{xo}e^{-\frac{z}{\delta}}\cos\left(\omega t - \frac{z}{\delta} - 45^o\right)$$

$H_y$ lags $E_x$ by $45^o$

**EXAMPLE 11.6**

Let us again consider wave propagation in water, but this time we will consider seawater. The primary difference between seawater and fresh water is of course the salt content. Sodium chloride dissociates in water to form $Na^+$ and $Cl^-$ ions, which, being charged, will move when forced by an electric field. Seawater is thus conductive, and so it will attenuate electromagnetic waves by this mechanism. At frequencies in the vicinity of $10^7$ Hz and below, the bound charge effects in water discussed earlier are negligible, and losses in seawater arise principally from the salt-associated conductivity. We consider an incident wave of frequency 1 MHz. We wish to find the skin depth, wavelength, and phase velocity. In seawater, $\sigma = 4$ S/m, and $\epsilon'_r = 81$.

**Solution.** We first evaluate the loss tangent, using the given data:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{(2\pi \times 10^6)(81)(8.85 \times 10^{-12})} = 8.9 \times 10^2 \gg 1$$

Seawater is therefore a good conductor at 1 MHz (and at frequencies lower than this). The skin depth is

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{(\pi \times 10^6)(4\pi \times 10^{-7})(4)}} = 0.25 \text{ m} = 25 \text{ cm}$$

Now

$$\lambda = 2\pi\delta = 1.6 \text{ m}$$

and

$$v_p = \omega\delta = (2\pi \times 10^6)(0.25) = 1.6 \times 10^6 \text{ m/sec}$$

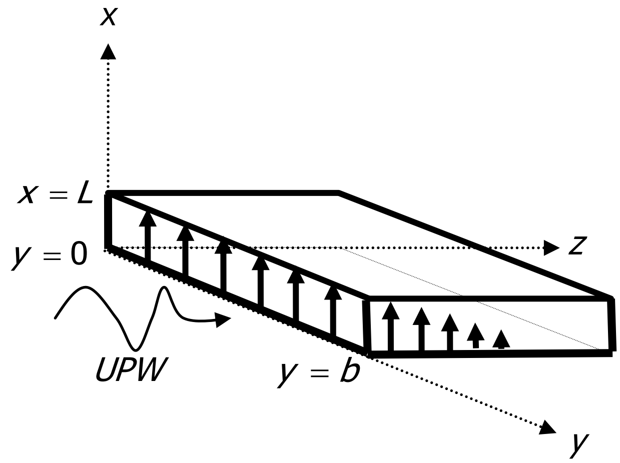
In free space, these values would have been $\lambda = 300$ m and of course $v = c$.

With a 25-cm skin depth, it is obvious that radio frequency communication in seawater is quite impractical. Notice, however, that $\delta$ varies as $1/\sqrt{f}$, so that things will improve at lower frequencies. For example, if we use a frequency of 10 Hz (in the ELF, or extremely low frequency range), the skin depth is increased over that at 1 MHz by a factor of $\sqrt{10^6/10}$, so that

$$\delta(10 \text{ Hz}) \doteq 80 \text{ m}$$

The corresponding wavelength is $\lambda = 2\pi\delta \doteq 500$ m. Frequencies in the ELF range were used for many years in submarine communications. Signals were transmitted from gigantic ground-based antennas (required because the free-space wavelength associated with 10 Hz is $3 \times 10^7$ m). The signals were then received by submarines, from which a suspended wire antenna of length shorter than 500 m is sufficient. The drawback is that signal data rates at ELF are slow enough that a single word can take several minutes to transmit. Typically, ELF signals would be used to tell the submarine to initiate emergency procedures, or to come near the surface in order to receive a more detailed message via satellite.

## Attacking the power problem (in good conductors);




$$E_x(z,t) = E_{xo} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$H_y(z,t) = \frac{\sigma\delta}{\sqrt{2}} E_{xo} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta} - 45^o\right)$$

$$J_x(z,t) = \sigma E_{xo} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$= J_{xo} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{\sigma\delta}{4} E_{xo}^2 e^{-\frac{2z}{\delta}} \hat{\mathbf{a}}_z$$

The total average power (loss) crossing the conductor surface at $z = 0$;

$$P_L = \iint_S \langle \mathbf{S} \rangle \bullet d\mathbf{s} = \int_{y=0}^{b} \int_{x=0}^{L} \frac{\sigma\delta}{4} E_{xo}^2 e^{-\frac{2z}{\delta}} \Big|_{z=0} \hat{\mathbf{a}}_z \bullet dxdy\hat{\mathbf{a}}_z$$

$$= \frac{\sigma\delta b L}{4} E_{xo}^2 = \frac{\delta b L}{4\sigma} J_{xo}^2$$

What result would be obtained for the power loss, if it is assumed that the total current is distributed **uniformly in one skin depth**.

To calculate the total current crossing the surface at $x = 0$, rewrite in phasor form:

$$J_{xs}(z) = J_{xo} e^{-(1+j)\frac{z}{\delta}}$$

So

$$I_s = \iint_S \mathbf{J} \bullet d\mathbf{s} = \int_{z=0}^{\infty} \int_{y=0}^{b} J_{xo} e^{-(1+j)\frac{z}{\delta}} dydz = \frac{J_{xo} b\delta}{1+j}$$

and

$$I(t) == \frac{J_{xo} b\delta}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

Assuming this current is distributed uniformly with current density $J_{uniform} = \frac{I}{S}$ through the cross section $S = b\delta$ then

$$J_{uniform} = \frac{I}{S} = \frac{J_{xo}}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

Then the total instantaneous power dissipated in volume of one skin depth thickness is:

$$P_{L_{ins}} = \iiint_V \mathbf{J}_{uniform} \bullet \mathbf{E} dv$$

$$= \int_{z=0}^{\delta} \int_{y=0}^{b} \int_{x=0}^{L} \frac{1}{\sigma} \left[\frac{J_{xo}}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)\right]^2 dxdydz$$

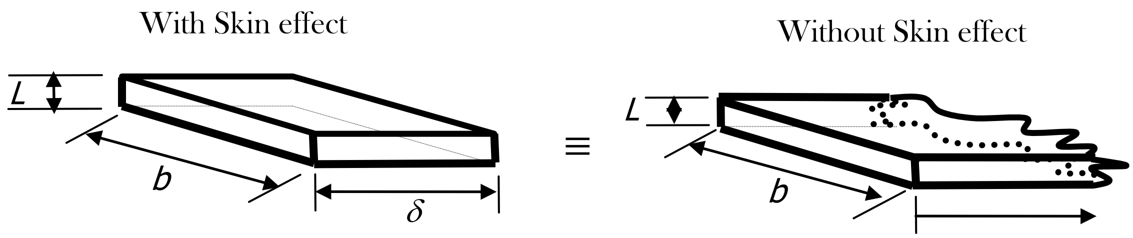
$$= \frac{J_{xo}^2}{2\sigma} bL\delta \cos^2\left(\omega t - \frac{\pi}{4}\right)$$

And the time average power loss within this volume is;

$$P_L = \int_T P_{L_{ins}} dt = \frac{J_{xo}^2}{4\sigma} bL\delta$$

This is exactly the same formula obtained before.

***Conclusion:*** *The average power loss in a conductor with skin effect may be calculated assuming that the total current is distributed uniformly in one skin depth. Or, the resistance of width b and length L of an infinitely thick slab with skin effect is the same as the resistance of a rectangular slab of width b, length L, and thickness $\delta$ without skin effect.*




$$R_{dc} = \frac{L}{\sigma S}$$

$$R_{ac} \approx \frac{L}{\sigma b \delta}$$

For a circular cross-section wire with radius $a >> \delta$, and at high frequencies

$$R = \frac{L}{\sigma S} \approx \frac{L}{2\pi a \sigma \delta}$$

More often, the ***surface or skin resistance*** $R_s$ $\left(in\ \Omega/_{m^2}\right)$ is defined as the real part of the intrinsic impedance for a good conductor. Thus

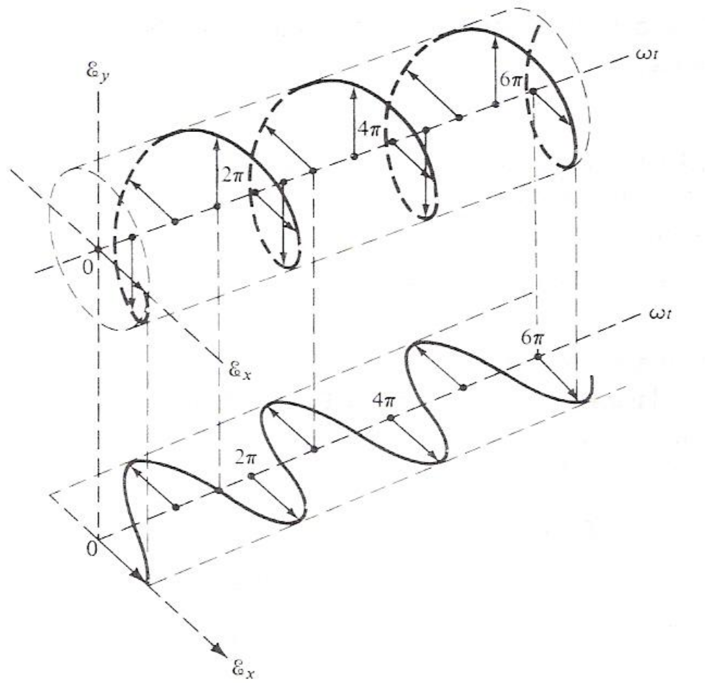
$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

since

$$\eta \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{1}{\sigma\delta}[1 + j1] = \frac{\sqrt{2}\angle 45^o}{\sigma\delta}$$

# WAVE POLARIZATION

**P**olarization is defined as *The locus that the tip of the* **E** *field traces as time varies for a fixed point in space. Or, the time-varying behavior of* **E** *at a given point in space.*




**C**onsider a wave propagating in the negative z-direction

$$\mathbf{E}(z,t) = \hat{\mathbf{a}}_x \mathrm{E}_x(z,t) + \hat{\mathbf{a}}_y \mathrm{E}_y(z,t)$$

Assume each component have a sinusoidal time dependence, and Since $\cos(\omega t + \theta) = \mathrm{Re}\left[e^{j(\omega t+\theta)}\right]$, then each component maybe written as a real part of some complex quantity (complex phasor);

$$\mathrm{E}_x(z,t) = \mathrm{Re}\left[\hat{E}_x(z) e^{j\omega t}\right]$$

with

$$\hat{E}_x(z) = E_x^- e^{jkz} = E_x e^{jkz} e^{j\theta_x} \quad \{\text{Complex phasor}$$

$$\mathrm{E}_y(z,t) = \mathrm{Re}\left[\hat{E}_y(z) e^{j\omega t}\right]$$

with

$$\hat{E}_y(z) = E_y^- e^{jkz} = E_y e^{jkz} e^{j\theta_y} \quad \{\text{Complex phasor}$$

So, the time domain representation is obtained as;

$$\mathrm{E}_x(z,t) = \mathrm{Re}\left[E_x e^{j(\omega t + kz + \theta_x)}\right] = E_x \cos(\omega t + kz + \theta_x)$$
$$\mathrm{E}_y(z,t) = \mathrm{Re}\left[E_y e^{j(\omega t + kz + \theta_y)}\right] = E_y \cos(\omega t + kz + \theta_y)$$

Three cases are to be considered.

**Case1: Linear polarization**

$$\Delta\theta \equiv \theta_y - \theta_x = n\pi \ \text{ with } n = 0,1,2,\ldots$$

**Example:** Find the polarization (linear, circular, elliptical) and sense of rotation for the uniform plane wave whose electric field is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{a}}_x 10\cos(\omega t + kz) + \hat{\mathbf{a}}_y 5\cos(\omega t + kz + \pi)$$

**Solution:**

Take $z = 0 \Rightarrow$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) + \hat{\mathbf{a}}_y 5\cos(\omega t + \pi)$$

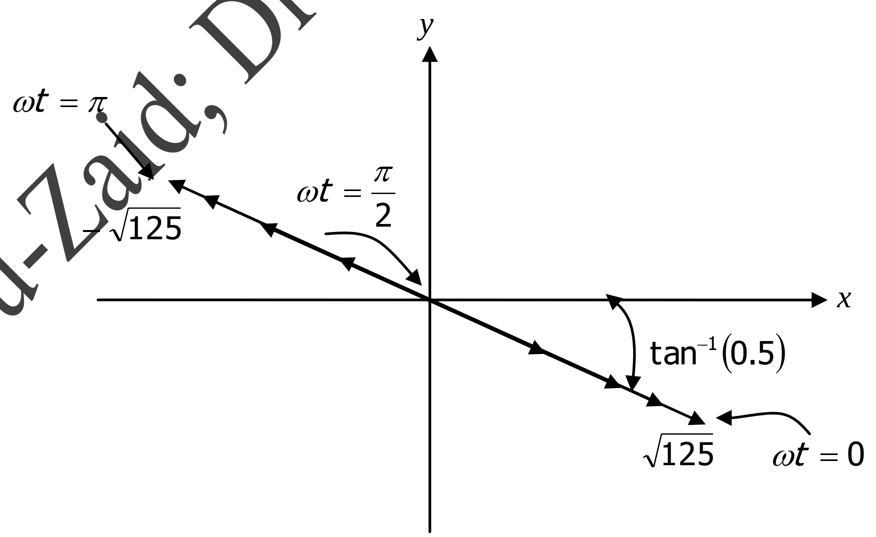
And since $\cos(\omega t) = \cos(\omega t + \pi)$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) + \hat{\mathbf{a}}_y 5\cos(\omega t)$$

$$|\mathbf{E}(0,t)| = \sqrt{10^2\cos^2(\omega t) + 5^2\cos^2(\omega t)}$$

$$= \sqrt{125\cos^2(\omega t)} = \sqrt{125}\cos(\omega t)$$

$$\angle\mathbf{E}(0,t) = \tan^{-1}\left(\frac{5\cos(\omega t)}{10\cos(\omega t)}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$



Linearly polarized with an angle of $\tan^{-1}(0.5) = 26.56^o$.
Note that $\theta_y = \pi$, $\theta_x = 0$ and $\Delta\theta = \theta_y - \theta_x = \pi - 0 = \pi$. So, from the beginning we may state that the polarization is linear, but with what angle?

## Case2: Circular polarization

$$E_x = E_y$$

$$\Delta\theta \equiv \theta_y - \theta_x = \begin{cases} \left(\frac{1}{2} + 2n\right)\pi & \text{for CW or RHCP} \\ -\left(\frac{1}{2} + 2n\right)\pi & \text{for CCW or LHCP} \end{cases}$$

$$n = 0,1,2,\ldots$$

*If the direction of propagation is in the positive z-direction, then the phases for CW and CCW must be reversed.*

**Example:** Find the polarization (linear, circular, elliptical) and sense of rotation for the uniform plane wave whose electric field is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{a}}_x 10\cos(\omega t + kz) + \hat{\mathbf{a}}_y 10\cos\left(\omega t + kz + \frac{5\pi}{2}\right)$$

**Solution:**

Take $z = 0 \Rightarrow$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) + \hat{\mathbf{a}}_y 10\cos\left(\omega t + \frac{5\pi}{2}\right)$$

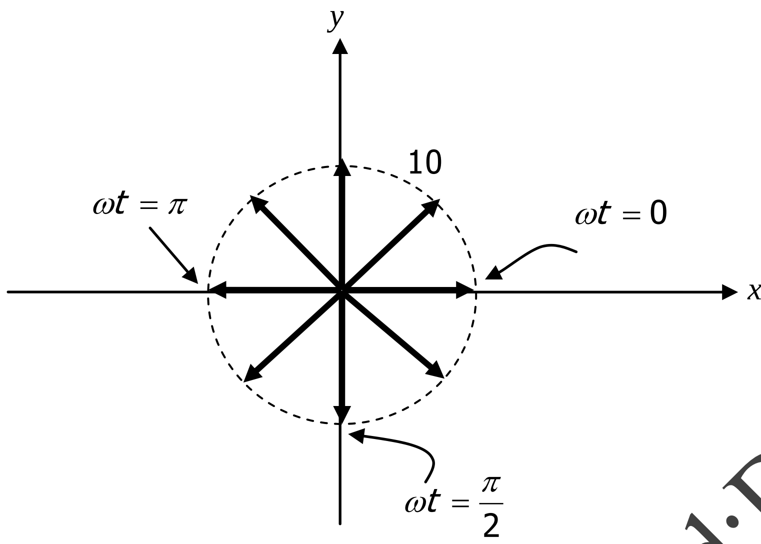
And since $\cos\left(\omega t + \frac{5\pi}{2}\right) = -\sin(\omega t)$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) - \hat{\mathbf{a}}_y 10\sin(\omega t)$$

$$|\mathbf{E}(0,t)| = \sqrt{10^2\cos^2(\omega t) + 10^2\sin^2(\omega t)}$$

$$= \sqrt{100\left(\cos^2(\omega t) + \sin^2(\omega t)\right)} = \sqrt{100} = 10$$

$$\angle\mathbf{E}(0,t) = \tan^{-1}\left(\frac{-10\sin(\omega t)}{10\cos(\omega t)}\right) = \tan^{-1}(-\tan(\omega t)) = -\omega t$$




Since the wave is propagating in negative z-direction, this is a RHCP circular polarization or CW polarization.

Note that:

$E_x = E_y = 10$, $\theta_y = \dfrac{5\pi}{2}$, $\theta_x = 0$

and

$$\Delta\theta = \theta_y - \theta_x = \frac{5\pi}{2} - 0 = \frac{5\pi}{2} = \left(\frac{1}{2} + 2\times 1\right)\pi.$$

So, from the beginning we may state that the polarization is CW circular.

## Case3: Elliptical polarization

$$E_x \neq E_y$$

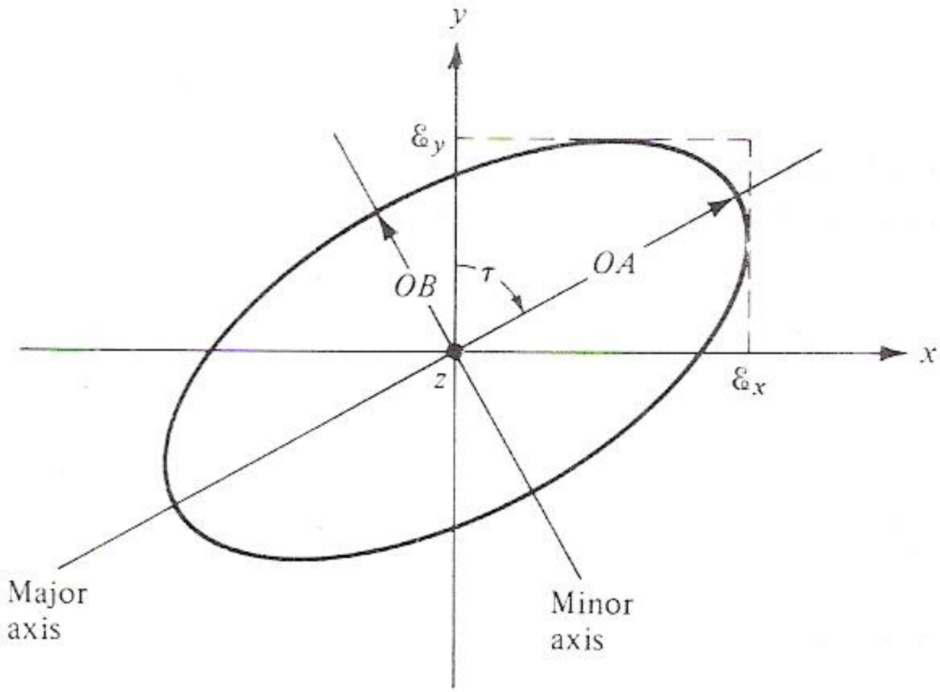
$$\Delta\theta \equiv \theta_y - \theta_x = \begin{cases} \left(\frac{1}{2} + 2n\right)\pi & \text{for CW or RHEP} \\ -\left(\frac{1}{2} + 2n\right)\pi & \text{for CCW or LHEP} \end{cases}$$

Or

$$\Delta\theta \equiv \theta_y - \theta_x \neq \pm\frac{n}{2}\pi = \begin{cases} > 0 & \text{for CW or RHEP} \\ < 0 & \text{for CCW or LHEP} \end{cases}$$

$$n = 0,1,2,\ldots$$

The curve traced is a tilted ellipse.






$$\text{Axial ratio} = AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{OA}{OB}$$

$$OA = \sqrt{\left[\frac{1}{2}\left\{E_x^2 + E_y^2 + \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\Delta\theta)}\right\}\right]}$$

$$OB = \sqrt{\left[\frac{1}{2}\left\{E_x^2 + E_y^2 - \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\Delta\theta)}\right\}\right]}$$

And the tilt angle w.r.t the y-axis is

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{2E_xE_y}{E_x^2 - E_y^2}\cos(\Delta\theta)\right]$$

**Example:** Find the polarization (linear, circular, elliptical), sense of rotation, axial ratio AR, and the tilt angle for the uniform plane wave whose electric field is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{a}}_x 10\cos(\omega t + kz) + \hat{\mathbf{a}}_y 0.5\cos\left(\omega t + kz + \frac{\pi}{2}\right)$$

**Solution:**

Take $z = 0 \Rightarrow$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) + \hat{\mathbf{a}}_y 5\cos\left(\omega t + \frac{\pi}{2}\right)$$

And since $\cos\left(\omega t + \frac{\pi}{2}\right) = -\sin(\omega t)$

$$\mathbf{E}(0,t) = \hat{\mathbf{a}}_x 10\cos(\omega t) - \hat{\mathbf{a}}_y 5\sin(\omega t)$$

$$|\mathbf{E}(0,t)| = \sqrt{10^2\cos^2(\omega t) + 5^2\sin^2(\omega t)}$$

$$\angle\mathbf{E}(0,t) = \tan^{-1}\left(\frac{-5\sin(\omega t)}{10\cos(\omega t)}\right) = \tan^{-1}(-0.5\tan(\omega t))$$

This is an elliptical polarization with

$$OA = \sqrt{\left[\frac{1}{2}\left\{E_x^2 + E_y^2 + \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\Delta\theta)}\right\}\right]}$$

$$= \sqrt{\left[\frac{1}{2}\left\{10^2 + 5^2 + \sqrt{10^4 + 5^4 + 2\times 10^2 \times 5^2\cos\left(2\frac{\pi}{2}\right)}\right\}\right]} = 10$$

$$OB = \sqrt{\left[\frac{1}{2}\left\{E_x^2 + E_y^2 - \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\Delta\theta)}\right\}\right]}$$

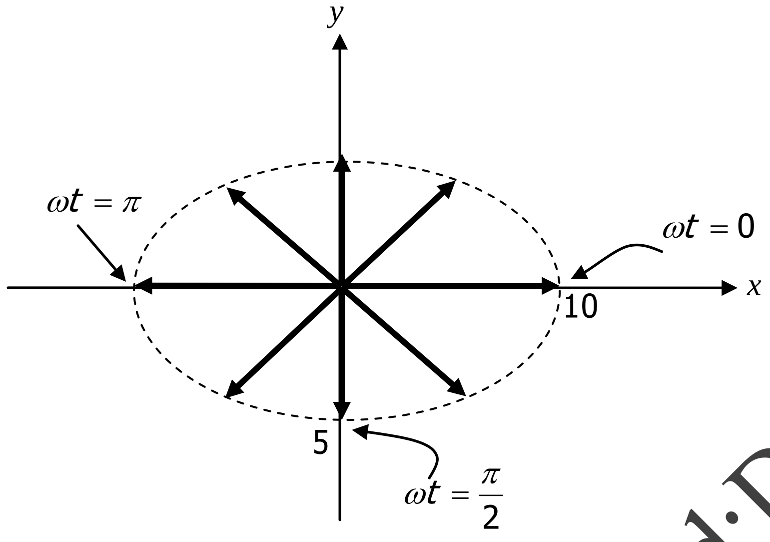
$$= \sqrt{\left[\frac{1}{2}\left\{10^2 + 5^2 - \sqrt{10^4 + 5^4 + 2\times 10^2 \times 5^2\cos\left(2\frac{\pi}{2}\right)}\right\}\right]} = 5$$

$$\text{Axial ratio} = AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{OA}{OB} = \frac{10}{5} = 2$$

And the tilt angle is

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{2E_xE_y}{E_x^2 - E_y^2}\cos(\Delta\theta)\right]$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{2\times 10\times 5}{10^2 - 5^2}\cos\left(\frac{\pi}{2}\right)\right] = \frac{\pi}{2}$$




Since the wave is propagating in negative z-direction, this is a RH elliptical polarization or CW elliptical polarization.

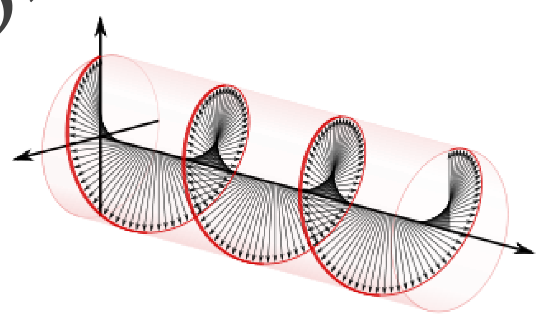
Note that:

$$E_x \neq E_y,\ \theta_y = \frac{\pi}{2},\ \theta_x = 0$$

and

$$\Delta\theta = \theta_y - \theta_x = \frac{\pi}{2} - 0 = \frac{\pi}{2} = \left(\frac{1}{2} + 2\times 0\right)\pi.$$

So, from the beginning we may state that the polarization is CW elliptical.



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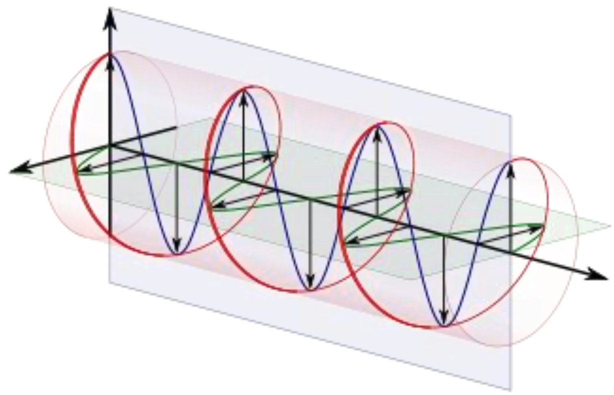
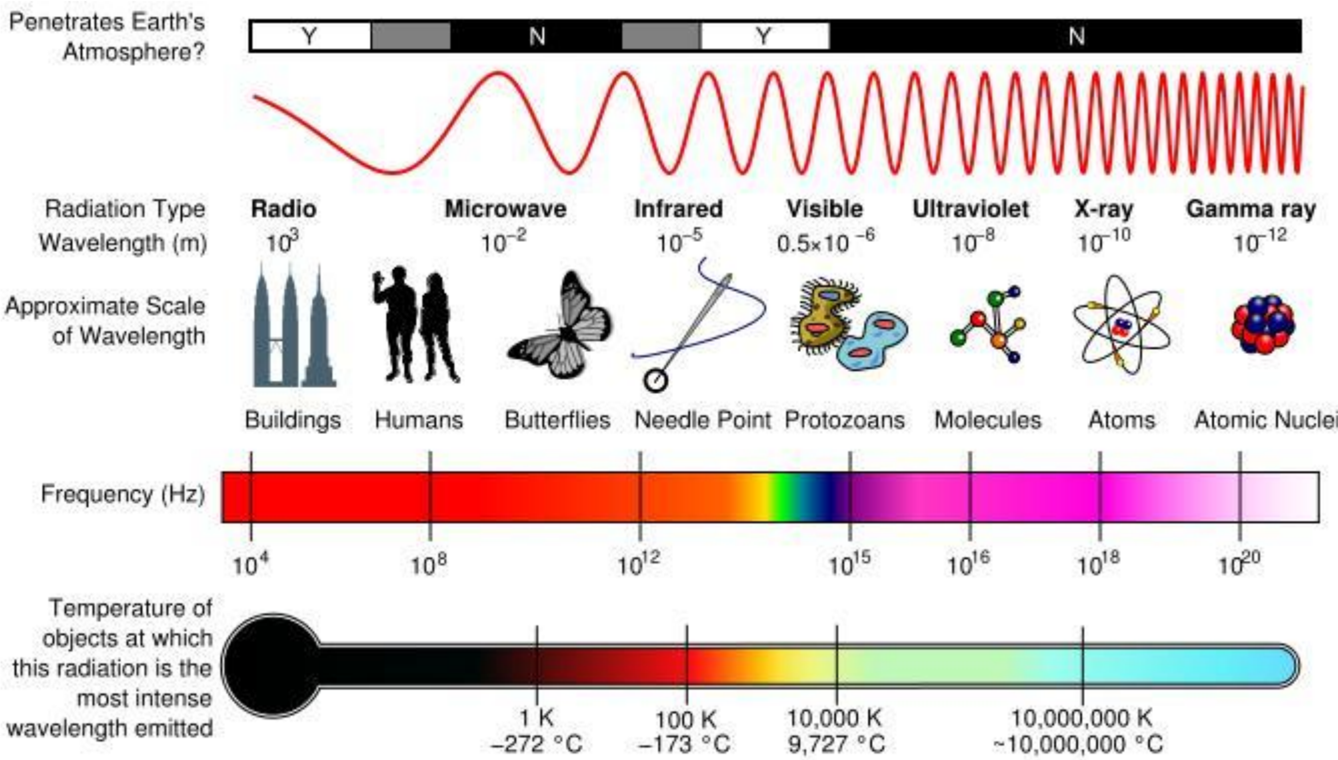


## THE ELECTROMAGNETIC SPECTRUM




$$f = \frac{c}{\lambda}, \quad \text{or} \quad f = \frac{E}{h}, \quad \text{or} \quad E = \frac{hc}{\lambda},$$

where:*c* = 299,792,458 m/s is the speed of light in vacuum and

- *h* = 6.62606896(33)×$10^{-34}$ J s = 4.13566733(10)×$10^{-15}$ eV s is Planck's constant.[5]

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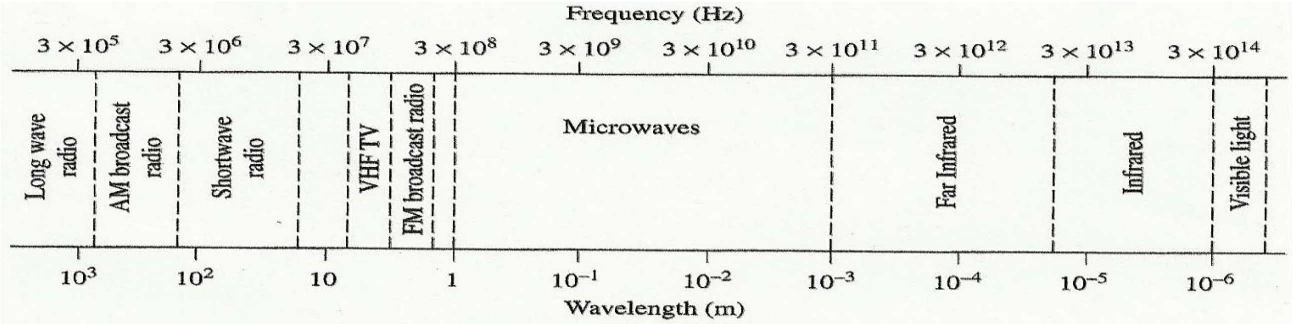


**Typical Frequencies**

| | |
|---|---|
| AM broadcast band | 535–1605 kHz |
| Short wave radio band | 3–30 MHz |
| FM broadcast band | 88–108 MHz |
| VHF TV (2–4) | 54–72 MHz |
| VHF TV (5–6) | 76–88 MHz |
| UHF TV (7–13) | 174–216 MHz |
| UHF TV (14–83) | 470–890 MHz |
| US cellular telephone | 824–849 MHz<br>869–894 MHz |
| European GSM cellular | 880–915 MHz<br>925–960 MHz |
| GPS | 1575.42 MHz<br>1227.60 MHz |
| Microwave ovens | 2.45 GHz |
| US DBS | 11.7–12.5 GHz |
| US ISM bands | 902–928 MHz<br>2.400–2.484 GHz<br>5.725–5.850 GHz |
| US UWB radio | 3.1–10.6 GHz |

**Approximate Band Designations**

| | |
|---|---|
| Medium frequency | 300 kHz to 3 MHz |
| High frequency (HF) | 3 MHz to 30 MHz |
| Very high frequency (VHF) | 30 MHz to 300 MHz |
| Ultra high frequency (UHF) | 300 MHz to 3 GHz |
| L band | 1–2 GHz |
| S band | 2–4 GHz |
| C band | 4–8 GHz |
| X band | 8–12 GHz |
| Ku band | 12–18 GHz |
| K band | 18–26 GHz |
| Ka band | 26–40 GHz |
| U band | 40–60 GHz |
| V band | 50-75 GHz |
| E band | 60-90 GHz |
| W band | 75-110 GHz |
| F band | 90-140 GHz |