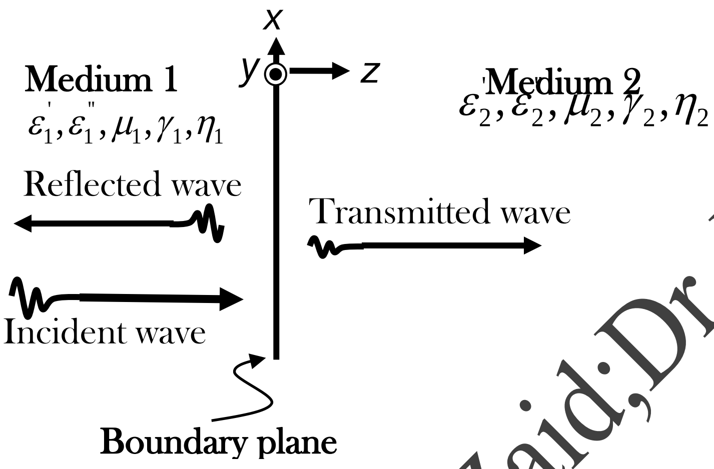
# Reflection and Transmission

# Chapter 13

# NORMAL INCIDENCE OF UPW ON PLANE BOUNDARIES




$$\gamma_1 = jk_1 = \alpha_1 + j\beta_1 = j\omega\sqrt{\mu_1\left(\varepsilon_1^{'} - j\varepsilon_2^{''}\right)}$$

$$\gamma_2 = jk_2 = \alpha_2 + j\beta_2 = j\omega\sqrt{\mu_2\left(\varepsilon_2^{'} - j\varepsilon_2^{''}\right)}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_1}{\varepsilon_1^{'} - j\varepsilon_1^{''}}} = |\eta_1|\angle\theta_{\eta_1}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_2}{\varepsilon_2^{'} - j\varepsilon_2^{''}}} = |\eta_2|\angle\theta_{\eta_2}$$

**Incident wave:**

Assume:

$$E_{x1}^{inc}(z,t) = E_{x10}^{inc} e^{-\alpha_1 z}\cos(\omega t - \beta_1 z)$$

$$E_{sx1}^{inc}(z) = E_{x10}^{inc}\, e^{-jk_1 z} = E_{x10}^{inc} e^{-\alpha_1 z}\, e^{-j\beta_1 z}$$

Then:

$$H_{ys1}^{inc}(z) = \frac{E_{x10}^{inc}}{\eta_1} e^{-jk_1 z}$$

**Transmitted wave:**

Assume:

$$E_{xs2}^{t}(z) = E_{x20}^{t}\, e^{-jk_2 z}$$

Then:

$$H_{ys2}^{t}(z)=\frac{E_{x20}^{t}}{\eta_2}e^{-jk_2 z}$$

Applying BC @ $z=0$, namely $E_{xs1}^{inc}(z)\Big|_{\substack{\tan\\ z=0}}=E_{xs2}^{t}(z)\Big|_{\substack{\tan\\ z=0}}$

$$\Rightarrow E_{x10}^{inc}=E_{x20}^{t}$$

$$H_{ys1}^{inc}(z)\Big|_{\substack{\tan\\ z=0}}=H_{ys2}^{t}(z)\Big|_{\substack{\tan\\ z=0}}$$

$$\Rightarrow \frac{E_{x10}^{inc}}{\eta_1}=\frac{E_{x20}^{t}}{\eta_2}\Rightarrow \eta_1=\eta_2 \text{ since } E_{x10}^{inc}=E_{x20}^{t}$$

So, it is a special case, and a ***reflected wave is required*** to satisfy the BC's in general.

**Reflected wave:**

Assume:

$$E_{sx1}^{r}(z)=E_{x10}^{r}e^{jk_1 z}$$

Then:

$$H_{ys1}^{r}(z)=-\frac{E_{x10}^{r}}{\eta_1}e^{jk_1 z}$$

Now, apply the BC to the total fields

Applying BC @ $z=0$, namely

$$E_{xs1}^{inc}(z)\Big|_{\substack{\tan\\ z=0}}+E_{xs1}^{r}(z)\Big|_{\substack{\tan\\ z=0}}=E_{xs2}^{t}(z)\Big|_{\substack{\tan\\ z=0}}$$

$$\Rightarrow E_{x10}^{inc}+E_{x10}^{r}=E_{x20}^{t} \quad \rightarrow (1)$$

$$H_{ys1}^{inc}(z)\Big|_{\substack{\tan\\ z=0}}+H_{ys1}^{r}(z)\Big|_{\substack{\tan\\ z=0}}=H_{ys2}^{t}(z)\Big|_{\substack{\tan\\ z=0}}$$

$$\Rightarrow \frac{E_{x10}^{inc}}{\eta_1}-\frac{E_{x10}^{r}}{\eta_1}=\frac{E_{x20}^{t}}{\eta_2} \quad \rightarrow (2)$$

Solving (1) and (2) for the ratios, namely

$$\underbrace{\Gamma}_{\text{Reflection Coefficient}}\equiv\frac{E_{x10}^{r}}{E_{x10}^{inc}}=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}=|\Gamma|e^{\theta_\Gamma}$$

$$\underbrace{\tau}_{\text{Transmission Coefficient}}\equiv\frac{E_{x20}^{t}}{E_{x10}^{inc}}=\frac{2\eta_2}{\eta_2+\eta_1}=1+\Gamma=|\tau|e^{\theta_\tau}$$

**Case 1:** Region 1 is a perfect Dielectric and region 2 is a perfect conductor $\sigma_2 \to \infty$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2^{'}}} = 0$$

$$\Rightarrow \tau = \frac{E_{x20}^{t}}{E_{x10}^{inc}} = \frac{2\eta_2}{\eta_2 + \eta_1} = 0 \Rightarrow E_{x20}^{t} = 0$$

and

$$\Gamma = \frac{E_{x10}^{r}}{E_{x10}^{inc}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1 \Rightarrow E_{x10}^{r} = -E_{x10}^{inc}$$

And the total field in region 1 is:

$$E_{sx1}^{total}(z) = E_{x10}^{inc}\, e^{-j\beta_1 z} - E_{x10}^{inc}\, e^{j\beta_1 z}$$
$$= -j2E_{x10}^{inc}\sin(\beta_1 z)$$

Since

$$\gamma_1 = jk_1 = \alpha_1 + j\beta_1 = j\beta_1 = j\omega\sqrt{\mu_1\varepsilon_1^{'}}$$

Converting to time domain

$$E_{x1}^{total}(z,t) = 2E_{x10}^{inc}\sin(\beta_1 z)\sin(\omega t)$$

The total field represents a standing wave.

**Time nulls** occurs when

$$\omega t = m\pi, \quad m = 0,1,2,3\ldots\ldots$$

**Space nulls** occurs when

$$\beta_1 z = m\pi, \quad m = 0,1,2,3\ldots\ldots$$

$$z = -m\frac{\lambda_1}{2}, \quad m = 0,1,2,3\ldots\ldots$$

The magnetic field, on the other hand is;

$$H_{ys1}^{total}(z) = H_{ys1}^{inc}(z) + H_{ys1}^{r}(z)$$
$$= \frac{E_{x10}^{inc}}{\eta_1} e^{-j\beta_1 z} + \frac{E_{x10}^{inc}}{\eta_1} e^{j\beta_1 z}$$

since

$$E_{x10}^{inc} = -E_{x10}^{r}$$

So;

$$H_{y1}^{total}(z,t) = 2\frac{E_{x10}^{inc}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

Again, a standing wave with maximums occurring whenever electric field is minimum. And the average power is zero.
The current induced on the surface of the perfect conductor is

$$\mathbf{K}_s = \hat{\mathbf{a}}_n \times \mathbf{H}_{ys1}^{total}(z)\Big|_{\substack{\tan\\ z=0}}$$

$$= -\hat{\mathbf{a}}_z \times \frac{E_{x10}^{inc}}{\eta_1}\left(e^{j\beta_1 z} + e^{-j\beta_1 z}\right)\hat{\mathbf{a}}_y\Bigg|_{\substack{\tan\\ z=0}}$$

$$= \hat{\mathbf{a}}_x \frac{2E_{x10}^{inc}}{\eta_1}$$

$$\mathbf{K} = \hat{\mathbf{a}}_x \frac{2E_{x10}^{inc}}{\eta_1}\cos(\omega t)$$

**Case 2:** Regions 1 and 2 are perfect dielectrics (lossless)
$$\left(\sigma_1 = \sigma_2 = 0 \rightarrow \varepsilon_1'' = \varepsilon_2'' = 0\right)$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_1}{\varepsilon_1'}} \text{ Real Positive}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_2}{\varepsilon_2'}} \text{ Real Positive}$$

$$\gamma_1 = jk_1 = \alpha_1 + j\beta_1 = j\omega\sqrt{\mu_1\varepsilon_1'}$$

$$\Rightarrow \alpha_1 = 0 \text{ and } \beta_1 = \omega\sqrt{\mu_1\varepsilon_1'}$$

$$\gamma_2 = jk_2 = \alpha_2 + j\beta_2 = j\omega\sqrt{\mu_2\varepsilon_2'}$$

$$\Rightarrow \alpha_2 = 0 \text{ and } \beta_2 = \omega\sqrt{\mu_2\varepsilon_2'}$$

$$\Rightarrow \tau = \frac{E_{x20}^{t}}{E_{x10}^{inc}} = \frac{2\eta_2}{\eta_2 + \eta_1} = |\tau|\angle 0^o$$

$$\Gamma = \frac{E_{x10}^{r}}{E_{x10}^{inc}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma|\angle 0^o \text{ or } |\Gamma|\angle 180^o$$

The fields are obtained as:

$$E_{sx1}^{inc}(z) = E_{x10}^{inc} e^{-j\beta_1 z}$$

$$H_{ys1}^{inc}(z) = \frac{E_{x10}^{inc}}{\eta_1} e^{-j\beta_1 z}$$

$$E_{sx1}^{r}(z) = E_{x10}^{r} e^{j\beta_1 z} = \Gamma E_{x10}^{inc} e^{j\beta_1 z}$$

$$H_{ys1}^{r}(z) = -\frac{E_{x10}^{r}}{\eta_1} e^{j\beta_1 z} = -\frac{\Gamma E_{x10}^{inc}}{\eta_1} e^{j\beta_1 z}$$

$$E_{xs2}^{t}(z) = E_{x20}^{t} e^{-j\beta_2 z} = \tau E_{x10}^{inc} e^{-j\beta_2 z}$$

$$H_{ys2}^{t}(z) = \frac{E_{x20}^{t}}{\eta_2} e^{-j\beta_2 z} = \frac{\tau E_{x10}^{inc}}{\eta_2} e^{-j\beta_2 z}$$

And the average power density of waves is

$$\left\langle \mathbf{S}_1^{inc} \right\rangle = \frac{1}{2}\mathrm{Re}\left\{ \mathbf{E}^{inc} \times \mathbf{H}^{inc^*} \right\} = \frac{\left|E_{x10}^{inc}\right|^2}{2\eta_1} \hat{\mathbf{a}}_z$$

$$\left\langle \mathbf{S}_1^{r} \right\rangle = \frac{1}{2}\mathrm{Re}\left\{ \mathbf{E}^{r} \times \mathbf{H}^{r^*} \right\} = -\frac{\left|E_{x10}^{inc}\right|^2 \left|\Gamma\right|^2}{2\eta_1} \hat{\mathbf{a}}_z$$

$$\Rightarrow \left|\left\langle \mathbf{S}_1^{r} \right\rangle\right| = \left|\Gamma\right|^2 \left|\left\langle \mathbf{S}_1^{inc} \right\rangle\right|$$

$$\left\langle \mathbf{S}_2^{t} \right\rangle = \frac{1}{2}\mathrm{Re}\left\{ \mathbf{E}^{t} \times \mathbf{H}^{t^*} \right\} = \frac{\left|E_{x10}^{inc}\right|^2 \left|\tau\right|^2}{2\eta_1} \hat{\mathbf{a}}_z$$

And taking energy conservation into account implies

$$\left|\left\langle \mathbf{S}_1^{inc} \right\rangle\right| = \left|\left\langle \mathbf{S}_1^{r} \right\rangle\right| + \left|\left\langle \mathbf{S}_2^{t} \right\rangle\right|$$

So,

$$\Rightarrow \left|\left\langle \mathbf{S}_2^{t} \right\rangle\right| = \left(1 - \left|\Gamma\right|^2\right) \left|\left\langle \mathbf{S}_1^{inc} \right\rangle\right|$$

## SWR

$$SWR = s = \frac{\left|E_1^{tot}(z)\right|_{\max}}{\left|E_1^{tot}(z)\right|_{\min}}$$

Assuming medium 1 is lossless $\alpha_1 = 0$;

$$E_{x1}^{tot}(z) = E_{x10}^{inc} e^{-j\beta_1 z} + \Gamma E_{x10}^{inc} e^{j\beta_1 z}$$

Then

$$E_{x1}^{tot}(z) = E_{x10}^{inc}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)$$

$$\text{since}\quad \Gamma = \frac{E_{x10}^{r}}{E_{x10}^{inc}} = |\Gamma| e^{j\theta_\Gamma}$$

The above expression may be written in time domain as:

$$E_{x1}^{tot}(z,t) = \underbrace{E_{x10}^{inc}(1-|\Gamma|)\cos(\omega t - \beta_1 z)}_{\text{Traveling part}}$$

$$+\underbrace{2E_{x10}^{inc}|\Gamma_L|\cos\left(\beta_1 Z + \frac{\theta_\Gamma}{2}\right)\cos\left(\omega t + \frac{\theta_\Gamma}{2}\right)}_{\text{Standing part}}$$

*Portion of the first incident wave reflects back and back propagates in region 1, and interferes with an equivalent portion of the 2nd incident wave to form a standing wave, the rest of the incident wave (which does not interfere) is the traveling wave part.*

**What is the voltage maximum and minimum and where do they occur?**

$$\begin{aligned} E_{x1}^{tot}(z) &= E_{x10}^{inc}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \\ &= E_{x10}^{inc}\left(e^{-j\beta_1 z} + |\Gamma| e^{j(\beta_1 z + \theta_\Gamma)}\right) \\ &= E_{x10}^{inc} e^{-j\beta z}\left(1 + |\Gamma| e^{j(2\beta_1 z + \theta_\Gamma)}\right) \end{aligned}$$

$$\left|E_{x1}^{tot}(z)\right| = \left|E_{x10}^{inc}\right| \left|1 + |\Gamma| e^{j(2\beta_1 z + \theta_\Gamma)}\right|$$

With a procedure similar to that done with TL's it can be shown that;
Maximum's occur when:

$$2\beta_1 z + \theta_\Gamma = -2m\pi, \quad m = 0,1,2,....$$

$$z_{\max} = \frac{-1}{2\beta_1}\left(2m\pi + \theta_\Gamma\right)$$

Then

$$\left|E_{x1}^{tot}(z)\right|_{\max} = \left|E_{x10}^{inc} e^{-j\beta z}\left(1 + |\Gamma| e^{j(2\beta_1 z + \theta_\Gamma)}\right)\right|_{z=z_{\max}}$$

$$= E_{x10}^{inc}\left(1 + |\Gamma|\right)$$

Minimum's occur when:

$$2\beta_1 z + \theta_\Gamma = -(2m+1)\pi, \quad m = 0,1,2,....$$

$$z_{\min} = \frac{-1}{2\beta_1}\left([2m+1]\pi + \theta_\Gamma\right)$$

Then

$$\left|E_{x1}^{tot}(z)\right|_{\min} = \left|E_{x10}^{inc}\, e^{-j\beta z}\left(1+|\Gamma| e^{j(2\beta_1 z+\theta_\Gamma)}\right)\right|_{z=z_{\min}}$$

$$= E_{x10}^{inc}\left(1-|\Gamma|\right)$$

And the SWR is obtained easily as:

$$SWR = s = \frac{\left|E_{x1}^{tot}(z)\right|_{\max}}{\left|E_{x1}^{tot}(z)\right|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$|\Gamma| = \frac{s-1}{s+1}$$

***Implication:*** $|\Gamma|$ *maybe found from measured* $s$*, and* $\theta_\Gamma$ *may be found from measured locations of maximum's and minimum's. Then the intrinsic impedance is known.*



**EXAMPLE 12.1**

As a numerical example we select

$$\eta_1 = 100\ \Omega$$
$$\eta_2 = 300\ \Omega$$
$$E_{x10}^{+} = 100\ \text{V/m}$$

and calculate values for the incident, reflected, and transmitted waves.

***Solution.*** The reflection coefficient is

$$\Gamma = \frac{300-100}{300+100} = 0.5$$

and thus

$$E_{x10}^{-} = 50\ \text{V/m}$$

The magnetic field intensities are

$$H_{y10}^{+} = \frac{100}{100} = 1.00 \text{ A/m}$$

$$H_{y10}^{-} = -\frac{50}{100} = -0.50 \text{ A/m}$$

Using Eq. (77) from Chapter 11, we find that the magnitude of the average incident power density is

$$\langle S_{1i} \rangle = \left| \frac{1}{2} \text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\} \right| = \frac{1}{2} E_{x10}^{+} H_{y10}^{+} = 50 \text{ W/m}^2$$

The average reflected power density is

$$\langle S_{1r} \rangle = -\frac{1}{2} E_{x10}^{-} H_{y10}^{-} = 12.5 \text{ W/m}^2$$

In region 2, using (10),

$$E_{x20}^{+} = \tau E_{x10}^{+} = 150 \text{ V/m}$$

and

$$H_{y20}^{+} = \frac{150}{300} = 0.500 \text{ A/m}$$

Therefore, the average power density that is transmitted through the boundary into region 2 is

$$\langle S_2 \rangle = \frac{1}{2} E_{x20}^{+} H_{y20}^{+} = 37.5 \text{ W/m}^2$$

We may check and confirm the power conservation requirement:

$$\langle S_{1i} \rangle = \langle S_{1r} \rangle + \langle S_2 \rangle$$

**EXAMPLE 12.2**

To illustrate some of these results, let us consider a 100-V/m, 3-GHz wave that is propagating in a material having $\epsilon'_{r1} = 4$, $\mu_{r1} = 1$, and $\epsilon''_r = 0$. The wave is normally incident on another perfect dielectric in region 2, $z > 0$, where $\epsilon'_{r2} = 9$ and $\mu_{r2} = 1$ (Figure 12.3). We seek the locations of the maxima and minima of **E**.

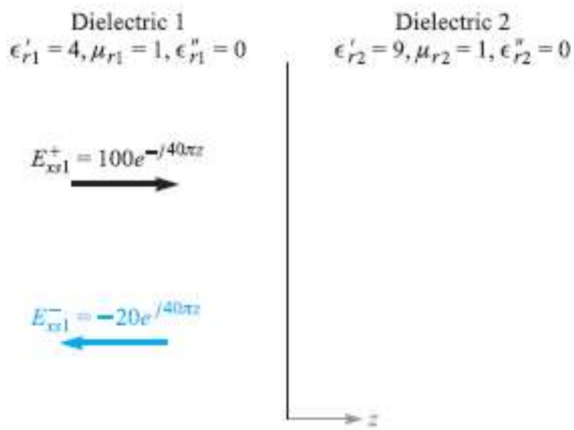



**Figure 12.3** An incident wave, $E^+_{xs1} = 100e^{-j40\pi z}$ V/m, is reflected with a reflection coefficient $\Gamma = -0.2$. Dielectric 2 is infinitely thick.

***Solution.*** We calculate $\omega = 6\pi \times 10^9$ rad/s, $\beta_1 = \omega\sqrt{\mu_1\epsilon_1} = 40\pi$ rad/m, and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2} = 60\pi$ rad/m. Although the wavelength would be 10 cm in air, we find here that $\lambda_1 = 2\pi/\beta_1 = 5$ cm, $\lambda_2 = 2\pi/\beta_2 = 3.33$ cm, $\eta_1 = 60\pi\ \Omega$, $\eta_2 = 40\pi\ \Omega$, and $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1) = -0.2$. Because $\Gamma$ is real and negative ($\eta_2 < \eta_1$), there will be a minimum of the electric field at the boundary, and it will be repeated at half-wavelength (2.5 cm) intervals in dielectric 1. From (23), we see that $|E_{x1T}|_{min} = 80$ V/m.

Maxima of **E** are found at distances of 1.25, 3.75, 6.25, ... cm from $z = 0$. These maxima all have amplitudes of 120 V/m, as predicted by (20).

There are no maxima or minima in region 2 because there is no reflected wave there.

**EXAMPLE 12.3**

A uniform plane wave in air partially reflects from the surface of a material whose properties are unknown. Measurements of the electric field in the region in front of the interface yield a 1.5-m spacing between maxima, with the first maximum occurring 0.75 m from the interface. A standing wave ratio of 5 is measured. Determine the intrinsic impedance, $\eta_u$, of the unknown material.

**Solution.** The 1.5 m spacing between maxima is $\lambda/2$, which implies that a wavelength is 3.0 m, or $f = 100$ MHz. The first maximum at 0.75 m is thus at a distance of $\lambda/4$ from the interface, which means that a field minimum occurs at the boundary. Thus $\Gamma$ will be real and negative. We use (27) to write

$$|\Gamma| = \frac{s-1}{s+1} = \frac{5-1}{5+1} = \frac{2}{3}$$

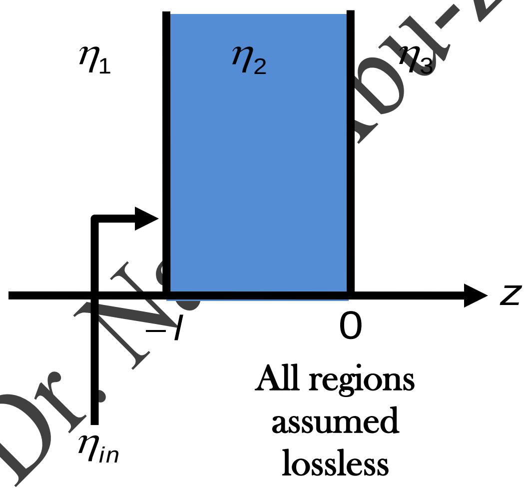
So

$$\Gamma = -\frac{2}{3} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

which we solve for $\eta_u$ to obtain

$$\eta_u = \frac{1}{5}\eta_0 = \frac{377}{5} = 75.4\,\Omega$$

## WAVE REFLECTIONS FROM MULTIPLE INTERFACES




$$E_{x2}^{tot}(z) = E_{x20}^{+} e^{-j\beta_2 z} + E_{x20}^{-} e^{j\beta_2 z}$$

$$H_{y2}^{tot}(z) = H_{y20}^{+} e^{-j\beta_2 z} + H_{y20}^{-} e^{j\beta_2 z}$$

$$\Gamma_{23} = \frac{E_{x20}^{-}}{E_{x20}^{+}} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

$$H_{y20}^{+} = \frac{E_{x20}^{+}}{\eta_2}$$

$$H_{y20}^{-} = -\frac{E_{x20}^{-}}{\eta_2} = -\Gamma_{23}\frac{E_{x20}^{+}}{\eta_2}$$

The **wave impedance** $\eta_w$ defined as the ratio of the total electric field to the total magnetic field.

In region 2, $\eta_w$ is given by

$$\eta_w(z) = \frac{E_{x2}^{tot}}{H_{y2}^{tot}} = \frac{E_{x20}^{+} e^{-j\beta_2 z} + E_{x20}^{-} e^{j\beta_2 z}}{H_{y20}^{+} e^{-j\beta_2 z} + H_{y20}^{-} e^{j\beta_2 z}}$$

$$= \eta_2 \left[ \frac{e^{-j\beta_2 z} + \Gamma_{23} e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23} e^{j\beta_2 z}} \right]$$

Using Euler's identities and the expression for $\Gamma_{23}$, $\eta_w$ may be written as

$$\eta_w(z) = \eta_2 \left[ \frac{\eta_3 \cos(\beta_2 z) - j\eta_2 \sin(\beta_2 z)}{\eta_2 \cos(\beta_2 z) - j\eta_3 \sin(\beta_2 z)} \right]$$

When evaluated at $z = -l$

$$\eta_{in} = \eta_2 \left[ \frac{\eta_3 \cos(\beta_2 l) + j\eta_2 \sin(\beta_2 l)}{\eta_2 \cos(\beta_2 l) + j\eta_3 \sin(\beta_2 l)} \right]$$

Using the wave impedance concept, and applying the BC's at the first interface $z = -l$, to solve the reflection problem and find the reflection coefficient

$$\Gamma = \frac{E_{x10}^{-}}{E_{x10}^{+}} = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1}$$

- ❖ $|\Gamma|^2$ Of the incident power is reflected from the interface.
- ❖ $\left(1 - |\Gamma|^2\right)$ of the incident power is transmitted into medium 3.

**Total transmission** $\{\Gamma = 0 \ when \ \eta_{in} = \eta_1\}$

Input impedance matched to incident medium.

$$\eta_2[\eta_3 \cos(\beta_2 l) + j\eta_2 \sin(\beta_2 l)] = \eta_1[\eta_2 \cos(\beta_2 l) + j\eta_3 \sin(\beta_2 l)]$$

$$\Rightarrow \ \eta_3 \cos(\beta_2 l) = \eta_1 \cos(\beta_2 l) \rightarrow (1)$$

$$and \quad \eta_2^2 \sin(\beta_2 l) = \eta_1 \eta_3 \sin(\beta_2 l) \rightarrow (2)$$

No reflection condition may be accomplished in two different ways:

**First way:** (half-wave matching)

$$\eta_3 = \eta_1 \ \ requires \quad l = m\frac{\lambda_2}{2}, m = 0,1,2.....$$

The ***refractive index*** of a lossless medium with $\mu_r = 1$ is defined as:

$$n = \frac{c}{v_p} = \sqrt{\varepsilon_r}$$

*n is defined as the factor by which the wavelength and the velocity of the radiation are reduced with respect to their vacuum values (or the ratio of the velocity of the wave in vacuum to its speed in media)*

Phase constant, intrinsic impedance, phase velocity, and wave length may be written in terms of refractive index for a lossless medium as:

$$\beta = \omega\sqrt{\varepsilon\mu} = \frac{n\omega}{c}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{n}$$

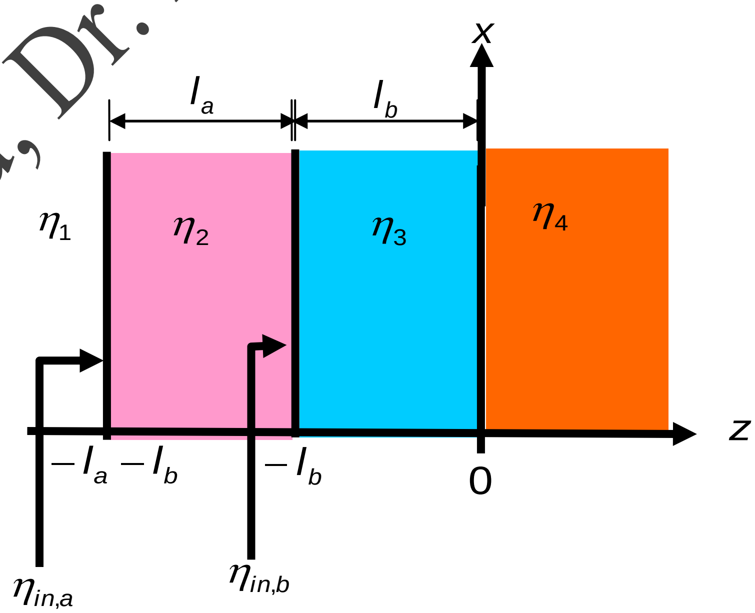
$$v_p = \frac{c}{n}$$

$$\lambda = \frac{v_p}{f} = \frac{\lambda_o}{n}$$

**Second way:** (Quarter wave matching)

$$\eta_3 \neq \eta_1 \; requires \;\; \eta_2 = \sqrt{\eta_1\eta_3}$$

$$l = (2m-1)\frac{\lambda_2}{4}, m = 1,2.....$$



**EXAMPLE 12.5**

We wish to coat a glass surface with an appropriate dielectric layer to provide total transmission from air to the glass at a free-space wavelength of 570 nm. The glass has refractive index $n_3 = 1.45$. Determine the required index for the coating and its minimum thickness.

**Solution.** The known impedances are $\eta_1 = 377\,\Omega$ and $\eta_3 = 377/1.45 = 260\,\Omega$. Using (46) we have

$$\eta_2 = \sqrt{(377)(260)} = 313\,\Omega$$

The index of region 2 will then be

$$n_2 = \left(\frac{377}{313}\right) = 1.20$$

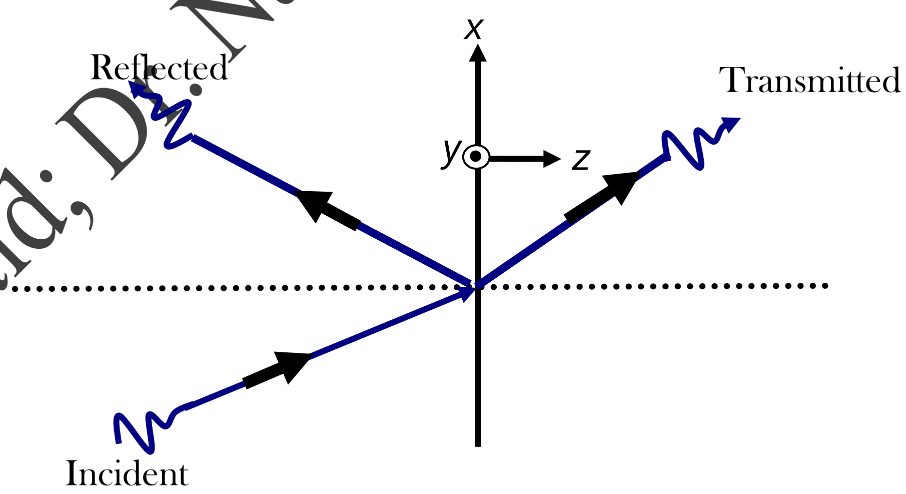
The wavelength in region 2 will be

$$\lambda_2 = \frac{570}{1.20} = 475\,\text{nm}$$

The minimum thickness of the dielectric layer is then

$$l = \frac{\lambda_2}{4} = 119\,\text{nm} = 0.119\,\mu\text{m}$$

## PLANE WAVE PROPAGATION IN ARBITRAY DIRECTIONS




How could we represent direction of propagation?
Considering propagation in a lossless media with propagation constant

$$\beta = k = \omega\sqrt{\varepsilon\mu}$$

The simplest case is the 2D one, where the wave propagates in a direction contained in the $xz$ plane.

Treating the ***propagation constant*** as a **vector** $\mathbf{k}$ **(Wave vector)**.(In the direction of power flow $\mathbf{S}$)

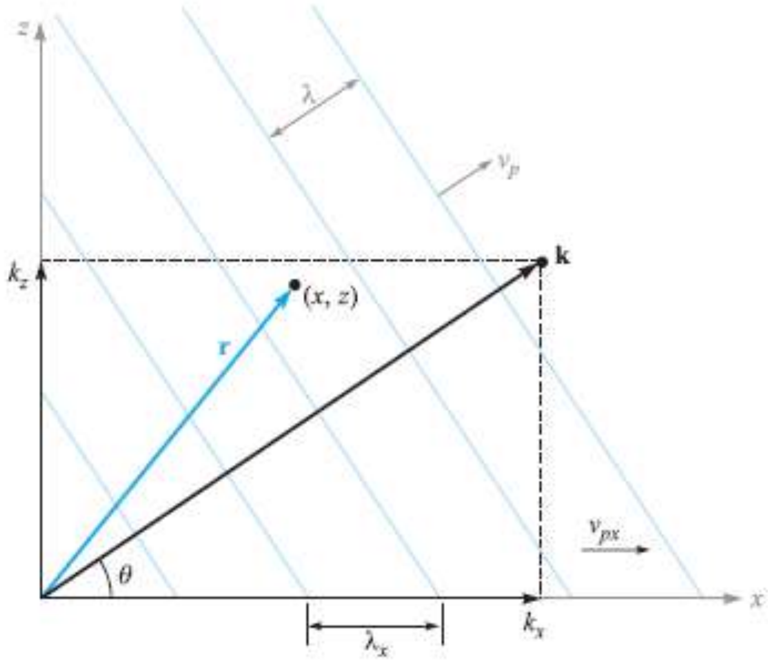



**Figure 12.6** Representation of a uniform plane wave with wavevector **k** at angle $\theta$ to the $x$ axis. The phase at point $(x, z)$ is given by $\mathbf{k}\cdot\mathbf{r}$. Planes of constant phase (shown as lines perpendicular to **k**) are spaced by wavelength $\lambda$ but have wider spacing when measured along the $x$ or $z$ axis.

**Phase fronts** (planes of constant phase).

$|\mathbf{k}|$ phase shift per unit distance along the direction of power flow.

To specify the phase of a plane wave, define the *radius vector* $\mathbf{r}$ (position vector) from the origin to a point in a plane of constant phase.

$$\mathbf{r} = x\,\hat{\mathbf{a}}_x + z\,\hat{\mathbf{a}}_y$$

Decomposing $\mathbf{k}$ into its components

$$\mathbf{k} = k\,\hat{\mathbf{a}}_n = k_x\hat{\mathbf{a}}_x + k_z\hat{\mathbf{a}}_z$$

The product

$$\mathbf{k}\bullet\mathbf{r} = xk_x + zk_z$$
$$= k\,\hat{\mathbf{a}}_n \bullet \mathbf{r}$$
$$= x\,k\,\hat{\mathbf{a}}_n \bullet \hat{\mathbf{a}}_x + z\,k\,\hat{\mathbf{a}}_n \bullet \hat{\mathbf{a}}_z$$
$$= x\,\underbrace{k\cos(\theta)}_{k_x} + z\,\underbrace{k\cos(90-\theta)}_{k_z}$$
$$\mathbf{k}\bullet\mathbf{r} = xk_x + zk_z$$
$$\tan(\theta) = \frac{k_z}{k_x}$$

So, the **E** and **H** of such plane wave maybe written as:

$$\mathbf{E} = \mathbf{E}_o e^{-j\mathbf{k}\bullet\mathbf{r}} = \mathbf{E}_o e^{-jk\hat{\mathbf{a}}_n\bullet\mathbf{r}}$$
$$= \mathbf{E}_o e^{-j(xk_x+zk_z)} = \underbrace{\mathbf{E}_o}_{\substack{\text{Constant}\\ \text{vector}\\ \text{forexample}\\ \mathrm{E_x\hat{a}_x+E_z\hat{a}_z}}} e^{-jxk\cos(\theta)-jzk\sin(\theta)}$$

$$\mathbf{H} = \mathbf{H}_o e^{-j\mathbf{k}\bullet\mathbf{r}} = \mathbf{H}_o e^{-jk\hat{\mathbf{a}}_n\bullet\mathbf{r}}$$
$$= \mathbf{H}_o e^{-j(xk_x+zk_z)} = \underbrace{\mathbf{H}_o}_{\substack{\text{Constant}\\ \text{vector}\\ \text{forexample}\\ \mathrm{E_y\hat{a}_y}}} e^{-jxk\cos(\theta)-jzk\sin(\theta)}$$

The other parameters in the direction of power flow are:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_z^2}}$$
$$v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{k_x^2 + k_z^2}}$$
$$f = \frac{v_p}{\lambda} = \frac{v_{px}}{\lambda_x}$$

**EXAMPLE 12.6**

Consider a 50-MHz uniform plane wave having electric field amplitude 10 V/m. The medium is lossless, having $\epsilon_r = \epsilon'_r = 9.0$ and $\mu_r = 1.0$. The wave propagates in the $x, y$ plane at a 30° angle to the $x$ axis and is linearly polarized along $z$. Write down the phasor expression for the electric field.

**Solution.** The propagation constant magnitude is

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega\sqrt{\epsilon_r}}{c} = \frac{2\pi \times 50 \times 10^6 (3)}{3 \times 10^8} = 3.2 \text{ m}^{-1}$$

The vector $\mathbf{k}$ is now

$$\mathbf{k} = 3.2(\cos 30\mathbf{a}_x + \sin 30\mathbf{a}_y) = 2.8\mathbf{a}_x + 1.6\mathbf{a}_y \text{ m}^{-1}$$

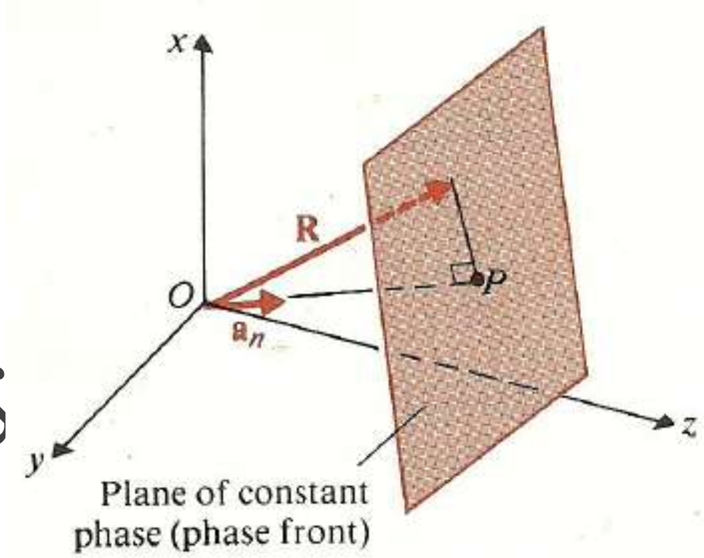
Then

$$\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y$$

With the electric field directed along $z$, the phasor form becomes

$$\mathbf{E}_s = E_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\,\mathbf{a}_z = 10e^{-j(2.8x+1.6y)}\,\mathbf{a}_z$$

## GENERALIZING TO 3D




Radius vector (position vector) and wave normal to a phase front of a uniform plane wave

**r**: The radius vector (position vector). In the graph this is shown as capital.

$$\mathbf{r} = x\hat{\mathbf{a}}_x + y\hat{\mathbf{a}}_y + z\hat{\mathbf{a}}_z$$

**k**: The propagation vector

$$\mathbf{k} = k\,\hat{\mathbf{a}}_n = k_x\hat{\mathbf{a}}_x + k_y\hat{\mathbf{a}}_y + k_z\hat{\mathbf{a}}_z$$

$$|\mathbf{k}| = k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

With $\hat{\mathbf{a}}_n$ as a unit vector in the direction of propagation (normal to the plane of constant phase)

$$\hat{\mathbf{a}}_n = \frac{\mathbf{k}}{|\mathbf{k}|}$$

The product which defines the phase (constant plane or phase front)

$$\begin{aligned}\mathbf{k} \bullet \mathbf{r} &= xk_x + yk_y + zk_z \\ &= k\,\hat{\mathbf{a}}_n \bullet \mathbf{r} \\ &= x\,k\,\hat{\mathbf{a}}_n \bullet \hat{\mathbf{a}}_x + y\,k\,\hat{\mathbf{a}}_n \bullet \hat{\mathbf{a}}_y + z\,k\,\hat{\mathbf{a}}_n \bullet \hat{\mathbf{a}}_z \\ &= x\underbrace{k\cos(\theta_x)}_{k_x} + y\underbrace{k\cos(\theta_y)}_{k_y} + z\underbrace{k\cos(\theta_z)}_{k_z}\end{aligned}$$

So, the **E** and **H** of such plane wave maybe written as:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_o e^{-j\mathbf{k}\bullet\mathbf{r}} = \mathbf{E}_o e^{-jk\hat{\mathbf{a}}_n\bullet\mathbf{r}} \\ &= \mathbf{E}_o e^{-j(xk_x+yk_y+zk_z)} \\ &= \underbrace{\mathbf{E}_o}_{\substack{\text{Constant}\\ \text{vector}}} e^{-jxk\cos(\theta_x)-jyk\cos(\theta_y)-jzk\cos(\theta_z)}\end{aligned}$$

$$\begin{aligned}\mathbf{H} &= \mathbf{H}_o e^{-j\mathbf{k}\bullet\mathbf{r}} = \mathbf{H}_o e^{-jk\hat{\mathbf{a}}_n\bullet\mathbf{r}} \\ &= \mathbf{H}_o e^{-j(xk_x+yk_y+zk_z)} \\ &= \underbrace{\mathbf{H}_o}_{\substack{\text{Constant}\\ \text{vector}}} e^{-jxk\cos(\theta_x)-jyk\cos(\theta_y)-jzk\cos(\theta_z)}\end{aligned}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$

$$v_p = \frac{\omega}{k} = \frac{\omega}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$

$$f = \frac{v_p}{\lambda}$$

**Note that:**

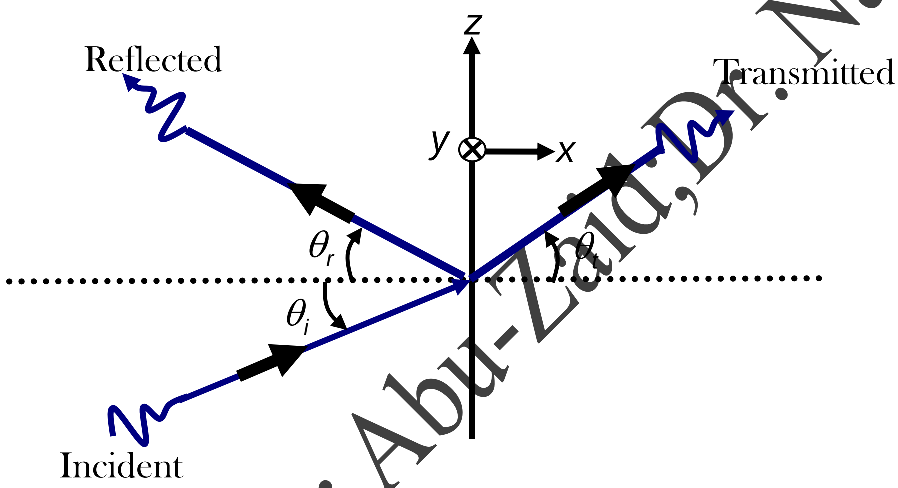
1) $\hat{\mathbf{a}}_n \bullet \mathbf{E}_o = 0 \quad \& \quad \mathbf{E}_o \perp \mathbf{H}_o \Rightarrow TEM$

2) $\mathbf{H} = \dfrac{\hat{\mathbf{a}}_n \times \mathbf{E}}{\eta} \quad or \quad \mathbf{E} = -\eta\,\hat{\mathbf{a}}_n \times \mathbf{H}$

# PW REFLECION AT OBLIQUE INCIDENC ANGLES

Objectives:

1) Determine relations between $\theta_i, \theta_r, \theta_t$.
2) Derive expressions for transmission and reflection coefficients.
3) Set the conditions required for total reflection or total transmission at the interface between two dielectrics.



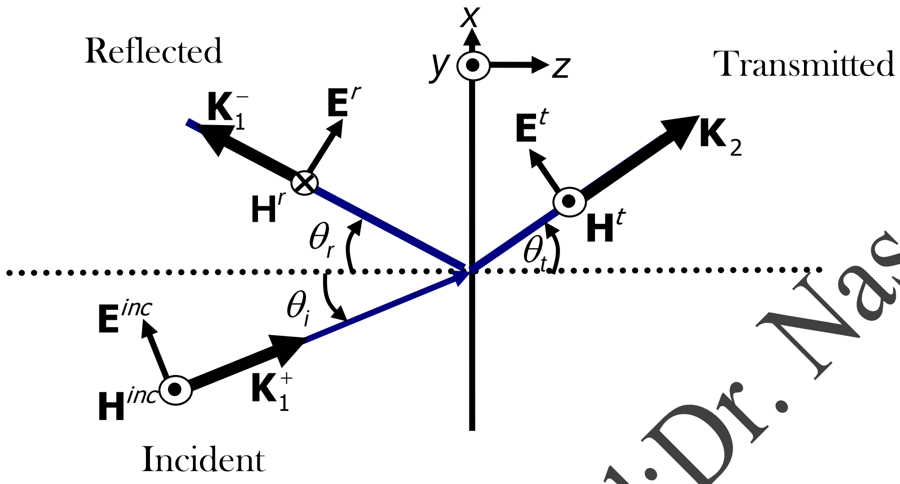

*Two lossless $\left(\varepsilon_r'' = 0\right)$ nonmagnetic materials $\left(\mu_o\right)$.*

***Plane of incidence:*** The plane containing the incident propagation vector **k** and the normal to the boundary surface.

***Parallel polarization*** or *p-polarization* or *TM polarization* (Transverse Magnetic since **H** is totally parallel to the boundary, or no component of $H$ is in the direction of propagation) all indicate the same situation where **E** *lies in the plane of incidence* (parallel to the plane of incidence).

***Perpendicular polarization*** or *s-polarization* or *TE polarization* (Transverse Electric since **E** is totally parallel to the boundary, or no component of $E$ is in the dirction of propagation) all indicate the same situation where **E** is perpendicular to the plane of incidence.
Other field directions can be decomposed into $s$ and $p$ waves.

## PARALLEL POLARIZATION




$$\mathbf{E}^{inc} = \mathbf{E}_{10}^{+} e^{-j\mathbf{k}_1^{+}\bullet\mathbf{r}}$$

$$\mathbf{E}^{r} = \mathbf{E}_{10}^{-} e^{-j\mathbf{k}_1^{-}\bullet\mathbf{r}}$$

$$\mathbf{E}^{t} = \mathbf{E}_{20} e^{-j\mathbf{k}_2\bullet\mathbf{r}}$$

With

$$\mathbf{k}_1^{+} = k_1\hat{\mathbf{a}}_{ni} = k_1\left[\cos(\theta_i)\hat{\mathbf{a}}_z + \sin(\theta_i)\hat{\mathbf{a}}_x\right]$$

$$\mathbf{k}_1^{-} = k_1\hat{\mathbf{a}}_{nr} = k_1\left[-\cos(\theta_r)\hat{\mathbf{a}}_z + \sin(\theta_r)\hat{\mathbf{a}}_x\right]$$

$$\mathbf{k}_2 = k_2\hat{\mathbf{a}}_{nt} = k_2\left[\cos(\theta_t)\hat{\mathbf{a}}_z + \sin(\theta_t)\hat{\mathbf{a}}_x\right]$$

$$\mathbf{r} = x\hat{\mathbf{a}}_x + z\hat{\mathbf{a}}_z$$

$$k_1 = \omega\sqrt{\varepsilon_1\mu_1} = \frac{\omega\sqrt{\varepsilon_{r1}}}{c} = \frac{\omega n_1}{c}$$

$$k_2 = \omega\sqrt{\varepsilon_2\mu_2} = \frac{\omega\sqrt{\varepsilon_{r2}}}{c} = \frac{\omega n_2}{c}$$

$$\mathbf{k}_1^{+}\bullet\mathbf{r} = k_1 z\cos(\theta_i) + k_1 x\sin(\theta_i)$$

$$\mathbf{k}_1^{-}\bullet\mathbf{r} = -k_1 z\cos(\theta_r) + k_1 x\sin(\theta_r)$$

$$\mathbf{k}_2\bullet\mathbf{r} = k_2 z\cos(\theta_t) + k_2 x\sin(\theta_t)$$

$$\mathbf{E}^{inc} = \mathbf{E}_{10}^{+} e^{-j\mathbf{k}_1^{+}\bullet\mathbf{r}}$$

$$= E_{10}^{+}\left[\hat{\mathbf{a}}_x\cos(\theta_i) - \hat{\mathbf{a}}_z\sin(\theta_i)\right]e^{-jk_1\left[x\sin(\theta_i)+z\cos(\theta_i)\right]}$$

$$\mathbf{H}^{inc} = \hat{\mathbf{a}}_y \frac{E_{10}^{+}}{\eta_1} e^{-jk_1[x\sin(\theta_i)+z\cos(\theta_i)]}$$

With the tangential component being

$$E_{x1}^{inc} = E_{10}^{+}\cos(\theta_i)e^{-jk_1[z\cos(\theta_i)+x\sin(\theta_i)]}$$

*You may try to find expressions for reflected and transmitted fields, both electric and magnetic!!!*

Applying BC @ $z=0$, namely:

$$E_{x1}^{inc}\Big|_{\substack{\tan\\ z=0}} + E_{x1}^{r}\Big|_{\substack{\tan\\ z=0}} = E_{x2}^{t}\Big|_{\substack{\tan\\ z=0}}$$

The above equation should hold for all values of $x$ (points on the interface), which implies that all phases must match, this establishes ***Snell's laws***:

$$\underbrace{k_1 x\sin(\theta_i) = k_1 x\sin(\theta_r)}_{\Rightarrow\theta_i=\theta_r} = k_2 x\sin(\theta_t)$$

Also the above equation requires that

$$\underbrace{\theta_i = \theta_r}_{\text{Snell's law of reflection}}$$

$$\underbrace{k_1\sin(\theta_i) = k_2\sin(\theta_t)}_{\text{Snell's law of refraction}}$$

And since $k = \frac{n\omega}{c}$, its usually written as;

$$n_1\sin(\theta_i) = n_2\sin(\theta_t)$$

Finding expressions for **H**, and applying the boundary conditions on tangential **H** @ $z=0$

$$H_{y1}^{inc}\Big|_{\substack{\tan\\ z=0}} + H_{y1}^{r}\Big|_{\substack{\tan\\ z=0}} = H_{y2}^{t}\Big|_{\substack{\tan\\ z=0}}$$

Accompanied with the boundary equation for **E** (which you should have found previously), and with the use of Snell's laws. One may obtain after some tedious mathematics;

$$\Gamma_p \equiv \frac{E_{10}^{r}}{E_{10}^{inc}} = \frac{\eta_{2p}-\eta_{1p}}{\eta_{2p}+\eta_{1p}}$$

$$\tau_p \equiv \frac{E_{20}^{t}}{E_{10}^{inc}} = \frac{2\eta_{2p}}{\eta_{2p}+\eta_{1p}}\left(\frac{\cos(\theta_i)}{\cos(\theta_t)}\right)$$

Where the effective impedances are defined as:

$$\eta_{1p} = \eta_1 \cos(\theta_i)$$

$$\eta_{2p} = \eta_2 \cos(\theta_t)$$

*The expression you were required to find:*

[redacted]

Now, apply the BC to the total fields. Applying BC @ [redacted] , namely;

[redacted]

since

[redacted]

For this equation to hold for all values of x, it is required that

[redacted]

or

[redacted]

Using the Boundary condition for the magnetic fields, and substituting snell's laws implies

[redacted]

Combining with the one for the electric field

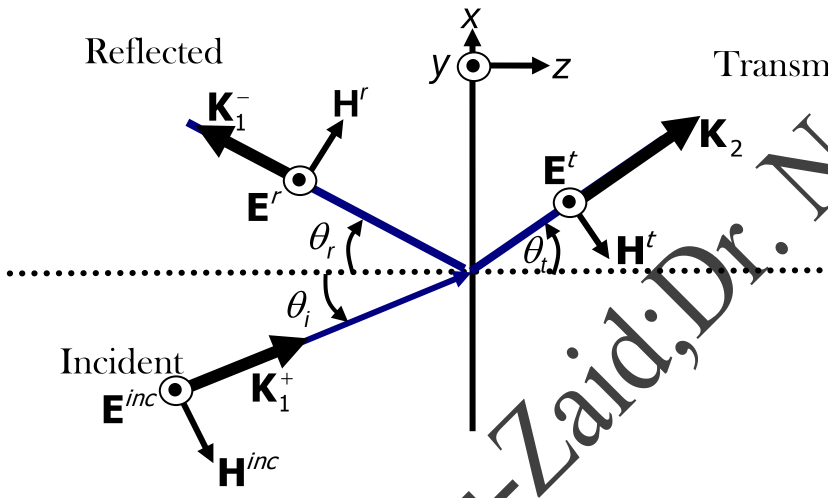
[redacted]

Solving (1) and (2) for the ratios

[redacted]

## PERPENDICULAR POLARIZATION

A similar formalism for perpendicular polarization with reference to the following figure may be accomplished with the results given as:




$$\Gamma_s \equiv \frac{E_{10}^r}{E_{10}^{inc}} = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}}$$

$$\tau_s \equiv \frac{E_{20}^t}{E_{10}^{inc}} = \frac{2\eta_{2s}}{\eta_{2s} + \eta_{1s}}$$

Where the effective impedances are defined as:

$$\eta_{1s} = \eta_1 \sec(\theta_i) = \frac{\eta_1}{\cos(\theta_i)}$$

$$\eta_{2s} = \eta_2 \sec(\theta_t) = \frac{\eta_2}{\cos(\theta_t)}$$

**EXAMPLE 12.7**

A uniform plane wave is incident from air onto glass at an angle from the normal of 30°. Determine the fraction of the incident power that is reflected and transmitted for (*a*) p-polarization and (*b*) s-polarization. Glass has refractive index $n_2 = 1.45$.

**Solution.** First, we apply Snell's law to find the transmission angle. Using $n_1 = 1$ for air, we use (63) to find

$$\theta_2 = \sin^{-1}\left(\frac{\sin 30}{1.45}\right) = 20.2^\circ$$

Now, for p-polarization:

$$\eta_{1p} = \eta_1 \cos 30 = (377)(.866) = 326\,\Omega$$

$$\eta_{2p} = \eta_2 \cos 20.2 = \frac{377}{1.45}(.938) = 244\,\Omega$$

Then, using (69), we find

$$\Gamma_p = \frac{244 - 326}{244 + 326} = -0.144$$

The fraction of the incident power that is reflected is

$$\frac{P_r}{P_{inc}} = |\Gamma_p|^2 = .021$$

The transmitted fraction is then

$$\frac{P_t}{P_{inc}} = 1 - |\Gamma_p|^2 = .979$$

For s-polarization, we have

$$\eta_{1s} = \eta_1 \sec 30 = 377/.866 = 435\,\Omega$$

$$\eta_{2s} = \eta_2 \sec 20.2 = \frac{377}{1.45(.938)} = 277\,\Omega$$

Then, using (71):

$$\Gamma_s = \frac{277 - 435}{277 + 435} = -.222$$

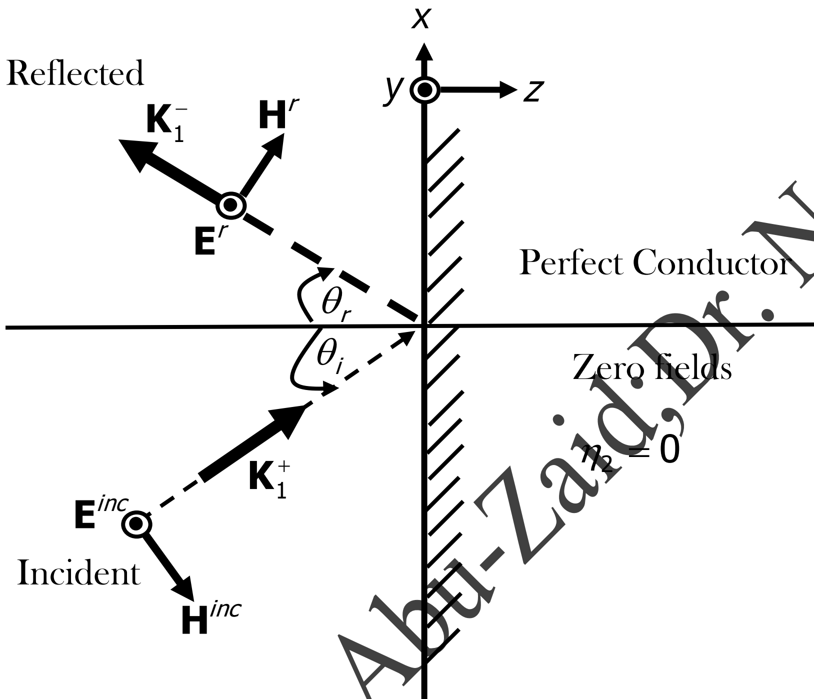
The reflected power fraction is thus

$$|\Gamma_s|^2 = .049$$

The fraction of the incident power that is transmitted is

$$1 - |\Gamma_s|^2 = .951$$

# OBLIQUE INCIDENCE ON PERFECT CONDUCTOR (PERPENDICULAR POLARIZATION)




$$\sigma_2 \to \infty \Rightarrow \eta_2 = 0 \Rightarrow \Gamma_s = -1 \Rightarrow E_{10}^+ = -E_{10}^-$$

$$\mathbf{E}^{tot} = E_{10}^+ \hat{\mathbf{a}}_y \, e^{-j k_1 [x \sin(\theta_i) + z \cos(\theta_i)]} - E_{10}^+ \hat{\mathbf{a}}_y \, e^{j k_1 [-x \sin(\theta_i) + z \cos(\theta_i)]}$$

Which may be written as:

$$\mathbf{E}_s^{tot} = -j \, \hat{\mathbf{a}}_y 2E_{10}^+ \, e^{-j k_1 x \sin(\theta_i)} \sin[k_1 z \cos(\theta_i)]$$

similarly for the magnetic field

$$\mathbf{H}^{tot} = -\hat{\mathbf{a}}_x \frac{2E_{10}^+}{\eta_1} e^{-j k_1 x \sin(\theta_i)} \cos(\theta_i) \cos[k_1 z \cos(\theta_i)] - \hat{\mathbf{a}}_z j \frac{2E_{10}^+}{\eta_1} e^{-j k_1 x \sin(\theta_i)} \sin(\theta_i) \sin[k_1 z \cos(\theta_i)]$$

and the time averaged power density is

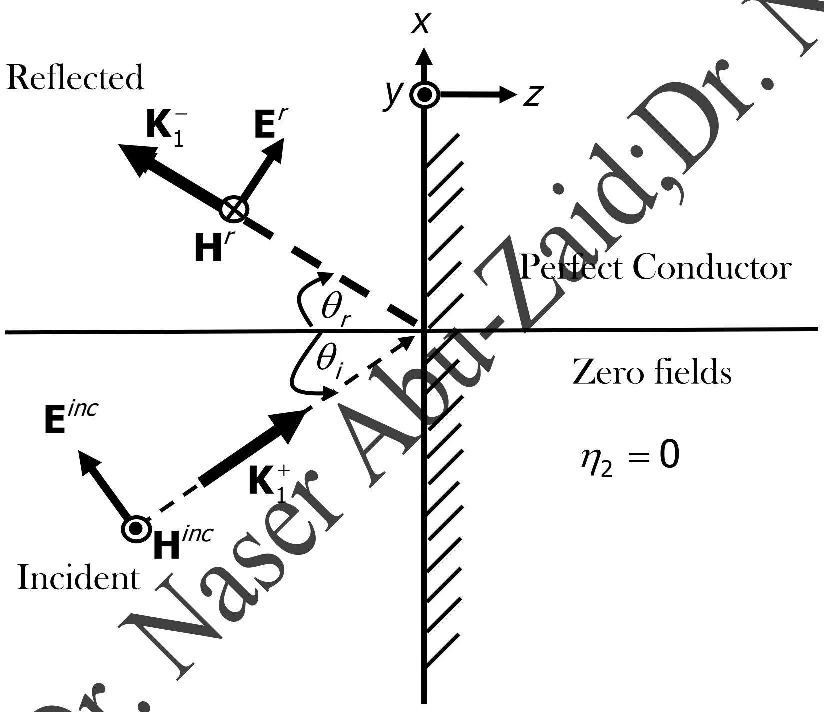
$$\langle \mathbf{S} \rangle = \frac{1}{2} \mathrm{Re}\left\{ \mathbf{E}^{tot} \times \mathbf{H}^{tot^*} \right\}$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{a}}_x \frac{2\left(E_{10}^{+}\right)^2}{\eta_1} \sin(\theta_i) \sin^2\left[k_1 z \cos(\theta_i)\right]$$

The surface current density is:

$$\mathbf{K} = \hat{\mathbf{a}}_n \times \mathbf{H}^{tot}\Big|_{z=0} = \hat{\mathbf{a}}_y \frac{2E_{10}^{+}}{\eta_1} \cos(\theta_i) \cos\left[\omega t - k_1 x \sin(\theta_i)\right]$$

## OBLIQUE INCIDENCE ON PERFECT CONDUCTOR (PARALLEL POLARIZATION)




$$\sigma_2 \to \infty \Rightarrow \eta_2 = 0 \Rightarrow \Gamma_p = -1 \Rightarrow E_{10}^{+} = -E_{10}^{-}$$

$$\begin{aligned}
\mathbf{E}^{tot} &= E_{10}^{+}\left[\hat{\mathbf{a}}_x \cos(\theta_i) - \hat{\mathbf{a}}_z \sin(\theta_i)\right] e^{-j k_1 [z\cos(\theta_i) + x\sin(\theta_i)]} \\
&- E_{10}^{+}\left[\hat{\mathbf{a}}_z \sin(\theta_i) + \hat{\mathbf{a}}_x \cos(\theta_i)\right] e^{-j k_1 [+x\sin(\theta_i) - z\cos(\theta_i)]} \\
&= -2E_{10}^{+}\begin{bmatrix} \hat{\mathbf{a}}_x j \cos(\theta_i) \sin(k_1 z \cos(\theta_i)) \\ + \hat{\mathbf{a}}_z \sin(\theta_i) \cos(k_1 z \cos(\theta_i)) \end{bmatrix} e^{-j k_1 x \sin(\theta_i)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}^{tot} &= \hat{\mathbf{a}}_y \frac{E_{10}^{+}}{\eta_1} e^{-j k_1 x \sin(\theta_i)} \left[e^{-j k_1 z \cos(\theta_i)} + e^{j k_1 z \cos(\theta_i)}\right] \\
&= \hat{\mathbf{a}}_y \frac{2E_{10}^{+}}{\eta_1} e^{-j k_1 x \sin(\theta_i)} \cos\left[k_1 z \cos(\theta_i)\right]
\end{aligned}$$

and the time averaged power density is

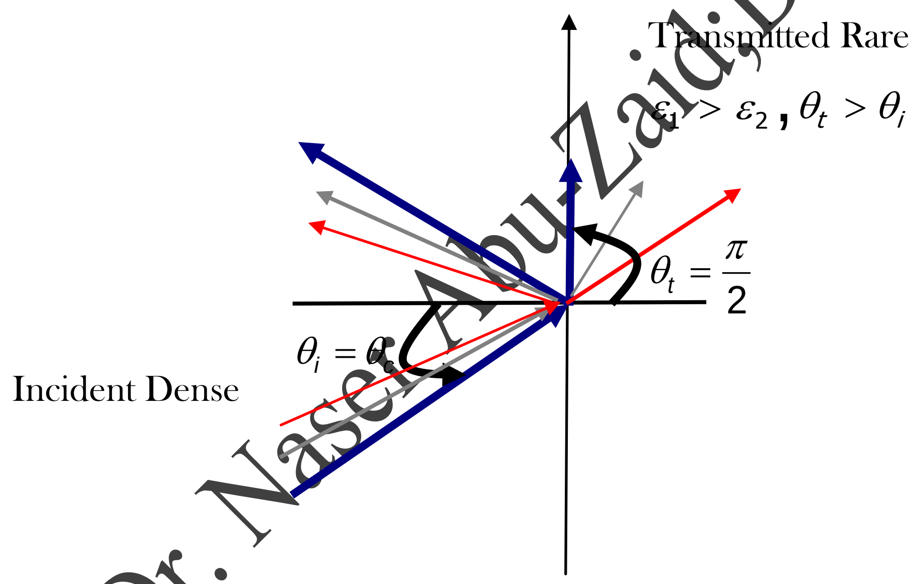
$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}^{tot} \times \mathbf{H}^{tot^*}\right\}$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{a}}_x \frac{2\left(E_{10}^{+}\right)^2}{\eta_1}\sin(\theta_i)\cos^2\left[k_1 z\cos(\theta_i)\right]$$

The surface current density is:

$$\mathbf{K} = \hat{\mathbf{a}}_n \times \mathbf{H}^{tot}\Big|_{z=0} = \hat{\mathbf{a}}_x \frac{2E_{10}^{+}}{\eta_1}\cos\left[\omega t - k_1 x\sin(\theta_i)\right]$$

## CRITICAL ANGLE OF TOTAL REFLECTION




$$|\Gamma|^2 = 1 \quad or \quad \Gamma\Gamma^* = 1 \quad \Rightarrow \quad \text{Total power reflection}$$

The obove condition stated in a different way is:

$$n_1\sin(\theta_i) = n_2\sin(\theta_t)$$

$$n_1\sin\left(\theta_i \geq \theta_c\right) = n_2\sin\left(\theta_t = \frac{\pi}{2}\right)$$

$$\frac{\sin(\theta_c)}{\sin\left(\theta_t = \frac{\pi}{2}\right)} = \frac{n_2}{n_1}$$

$$\underbrace{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)}_{\text{Any incidence angle } \theta_i \text{ greater than critical angle } \theta_c \text{ will cause total reflection}}$$

**Example 12.8:** A prism is used to turn a beam of light by $90^o$ as shown in the figure. Light enters and exists the prism through two antireflective (AR-coated) surfaces. Total reflection is to occur at the back surface, where the incident angle is $45^o$ to the normal. Determine the minimum required refractive index of the prism material if the surrounding region is air.

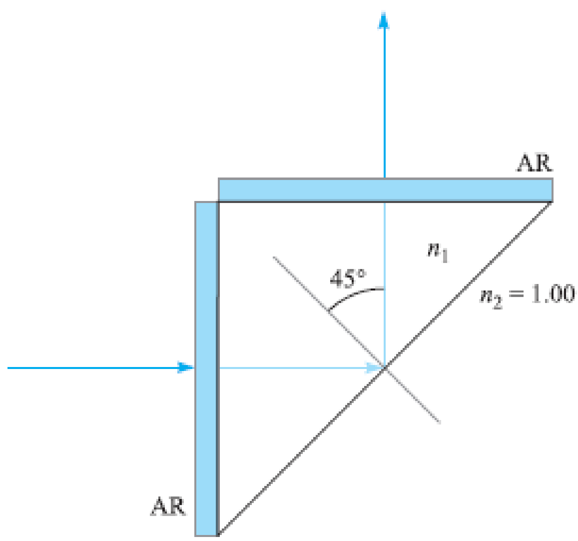



**Figure 12.8** Beam-steering prism for Example 12.8.

***Solution.*** Considering the back surface, the medium beyond the interface is air, with $n_2 = 1.00$. Because $\theta_1 = 45°$, (76) is used to obtain

$$n_1 \geq \frac{n_2}{\sin 45} = \sqrt{2} = 1.41$$

Because fused silica glass has refractive index $n_g = 1.45$, it is a suitable material for this application and is in fact widely used.

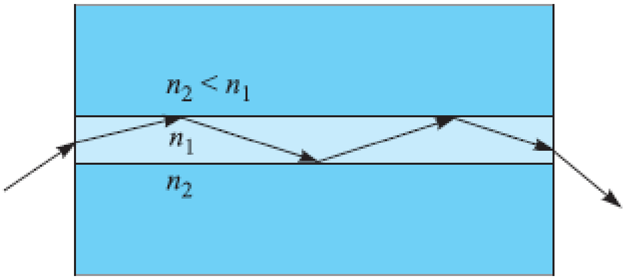



**Figure 12.9** A dielectric slab waveguide (symmetric case), showing light confinement to the center material by total reflection.

*Another important application of total reflection is in optical waveguides.*

## BREWSTER ANGLE OF TOTAL TRANSMISSION (OR POLARIZING ANGLE)

### For perpendicular polarization:

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}}$$

$$\eta_{1s} = \eta_1 \sec(\theta_i) \qquad \eta_{2s} = \eta_2 \sec(\theta_t)$$

$$\eta_1 \sec(\theta_i) = \eta_2 \sec(\theta_t)$$

The incidence angle required to satisfy the above condition is called *angle of total transmission or Brewester Angle* and is given by:

$$\underbrace{\theta_{B\perp} = \sin^{-1}\sqrt{\frac{1 - \frac{\mu_1 \varepsilon_2}{\mu_2 \varepsilon_1}}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}}}_{\text{An incidence angle } \theta_i \text{ satisfying the above condition will cause total transmisssion}}$$

For non-magnetic materials, Brewester angle for perpendicular polarization does not exist.

### For parallel polarization:

$$\Gamma_p = \frac{\eta_{2p} - \eta_{1p}}{\eta_{2p} + \eta_{1p}}$$

$$\eta_{1p} = \eta_1 \cos(\theta_i), \qquad \eta_{2p} = \eta_2 \cos(\theta_t)$$

$$\eta_1 \cos(\theta_i) = \eta_2 \cos(\theta_t)$$

The incidence angle required to satisfy the above condition is called *angle of total transmission or Brewester Angle* and given by:

$$\theta_{B\parallel} = \underbrace{\sin^{-1}\sqrt{\frac{1-\dfrac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1-\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2}}}_{\text{An incidence angle } \theta_i \text{ satisfying the above condition will cause total transmisssion}}$$

***The sum of incident and refracted angles under brewester condition is always $90^o$.***

**Example:** Light is incident from air to glass at Brewster's angle. Determine the incident and transmitted angles. Glass has a $n_2 = 1.45$.

***Solution.*** Because glass has refractive index $n_2 = 1.45$, the incident angle will be

$$\theta_1 = \theta_B = \sin^{-1}\left(\frac{n_2}{\sqrt{n_1^2+n_2^2}}\right) = \sin^{-1}\left(\frac{1.45}{\sqrt{1.45^2+1}}\right) = 55.4^\circ$$

The transmitted angle is found from Snell's law, through

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_B\right) = \sin^{-1}\left(\frac{n_1}{\sqrt{n_1^2+n_2^2}}\right) = 34.6^\circ$$

Note from this exercise that $\sin\theta_2 = \cos\theta_B$, which means that the sum of the incident and refracted angles at the Brewster condition is always 90°.

# DISPERSION

**Dispersion** is the phenomenon in which the ***phase velocity*** of a wave depends on its frequency, or alternatively when the ***group velocity*** depends on the frequency. The result is the distortion of signals transmitted through dispersive media.

Media having such a property are termed ***dispersive media***. Dispersion is most often described for ***light*** waves, but it may occur for any kind of wave that interacts with a medium or passes through an inhomogeneous geometry (e.g., a waveguide).

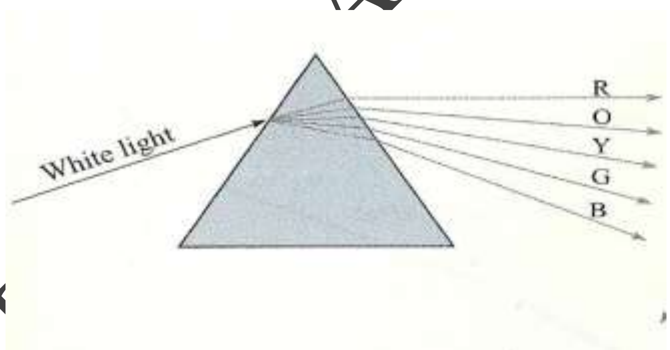
- *Non linear dependence* of $\beta$ on $\omega$ cause distortion of signals, such phenomenon is called dispersion.

$$\varepsilon_r(\omega) \rightarrow n(\omega) \rightarrow \beta(\omega)$$

- Dispersion: *Separation* of distinguishable components of a wave.
- For a lossless, non-magnetic media with $n(\omega)$ *varying with frequency*:

$$\beta(\omega) = \omega\sqrt{\mu_o \varepsilon(\omega)} = \omega\sqrt{\mu_o \varepsilon_o \varepsilon_r(\omega)} = \frac{\omega}{c} n(\omega)$$

- *Prism* frequency dependent refractive index results in different angles of refraction for *different frequencies (colors)*. Wave dispersion in space.




Consider the superposition of two x-polarized waves with different frequencies $\omega_a$ and $\omega_b$, both propagating in z-direction

$$\mathbf{E}_c = \hat{\mathbf{a}}_x E_m \left[ e^{-j\beta_a z} e^{j\omega_a t} + e^{-j\beta_b z} e^{j\omega_b t} \right]$$

$$\Delta\omega = \omega_o - \omega_a = \omega_b - \omega_o$$

$$\Delta\beta = \beta_o - \beta_a \approx \beta_b - \beta_o$$

Also assume $\Delta\omega << \omega_o$.

With some mathematical manipulations, this wave may be written as:

$$\mathbf{E}(z,t) = \hat{\mathbf{a}}_x E_m \underbrace{\cos(\Delta\omega t - \Delta\beta z)}_{\substack{\text{Low frequency wave} \\ @\Delta\omega \\ \text{(modulating wave)}}} \underbrace{\cos(\omega_o t - \beta_o z)}_{\substack{\textit{High Frequency wave} \\ @\omega_o \\ (\textit{Carrier})}}$$

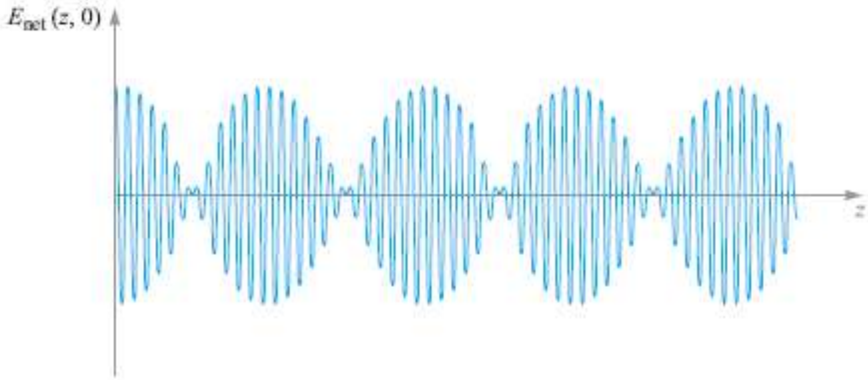



**Figure 12.13** Plot of the total electric field strength as a function of $z$ (with $t = 0$) of two co-propagating waves having different frequencies, $\omega_a$ and $\omega_b$, as per Eq. (81). The rapid oscillations are associated with the carrier frequency, $\omega_0 = (\omega_a + \omega_b)/2$. The slower modulation is associated with the envelope or "beat" frequency, $\Delta\omega = (\omega_b - \omega_a)/2$.

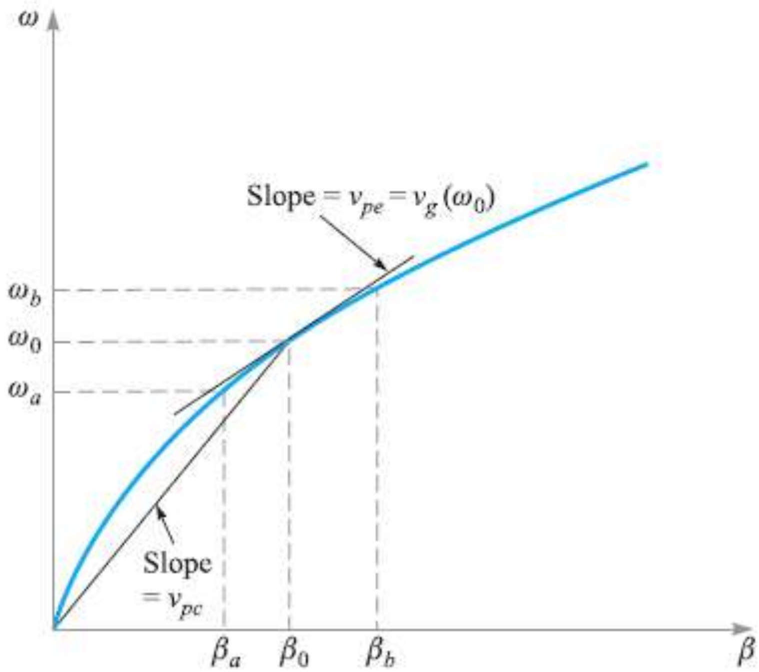



**Figure 12.12** ω-β diagram for a material in which the refractive index increases with frequency. The slope of a line tangent to the curve at $\omega_0$ is the group velocity at that frequency. The slope of a line joining the origin to the point on the curve at $\omega_0$ is the phase velocity at $\omega_0$.

*Carrier velocity (or phase velocity)*, wave inside envelop obtained by assuming constant carrier phase, and *envelope velocity (or group velocity)* obtained by assuming constant modulation phase.

$$v_{pc} = \frac{\omega_o}{\beta_o} \text{ (Phase velocity)}$$

$$v_{ge} = \frac{\Delta\omega}{\Delta\beta} \text{ (Group velocity)}$$

Letting $\omega_o \to \omega$ (variable frequency) and $\Delta\omega \to d\omega \Rightarrow \Delta\beta \to d\beta$ (separation between frequency components). One may obtain

$$v_p = \frac{\omega}{\beta}$$

$$v_g = \frac{d\omega}{d\beta}$$

Or

$$\lim_{\Delta\omega \to 0} \frac{\Delta\omega}{\Delta\beta} = \left.\frac{d\omega}{d\beta}\right|_{\omega_0} = v_g(\omega_0)$$

**EXAMPLE 12.10**

Consider a medium in which the refractive index varies linearly with frequency over a certain range:

$$n(\omega) = n_0 \frac{\omega}{\omega_0}$$

Determine the group velocity and the phase velocity of a wave at frequency $\omega_0$.

***Solution.*** First, the phase constant will be

$$\beta(\omega) = n(\omega)\frac{\omega}{c} = \frac{n_0 \omega^2}{\omega_0 c}$$

Now

$$\frac{d\beta}{d\omega} = \frac{2n_0\omega}{\omega_0 c}$$

so that

$$v_g = \frac{d\omega}{d\beta} = \frac{\omega_0 c}{2n_0\omega}$$

The group velocity at $\omega_0$ is

$$v_g(\omega_0) = \frac{c}{2n_0}$$

The phase velocity at $\omega_0$ will be

$$v_p(\omega_0) = \frac{\omega}{\beta(\omega_0)} = \frac{c}{n_0}$$

## PULSE BROADENING

- Dispersive media distorts pulses by broadening them in time.
- Consider a Gaussian pulse in time with an $\mathbf{E}$ field at $z = 0$ given as:

$$E(0,t) = E_m e^{-0.5\left(\frac{t}{T}\right)^2} e^{j\omega_o t}$$

$T$ : ***Characteristic half width*** (time at which the magnitude of the Poynting vector fall to $1/e$ of its maximum value.

The *frequency spectrum* is given by:

$$E(0,\omega) = \frac{E_m T}{\sqrt{2\pi}} e^{-0.5T^2(\omega-\omega_o)^2}$$

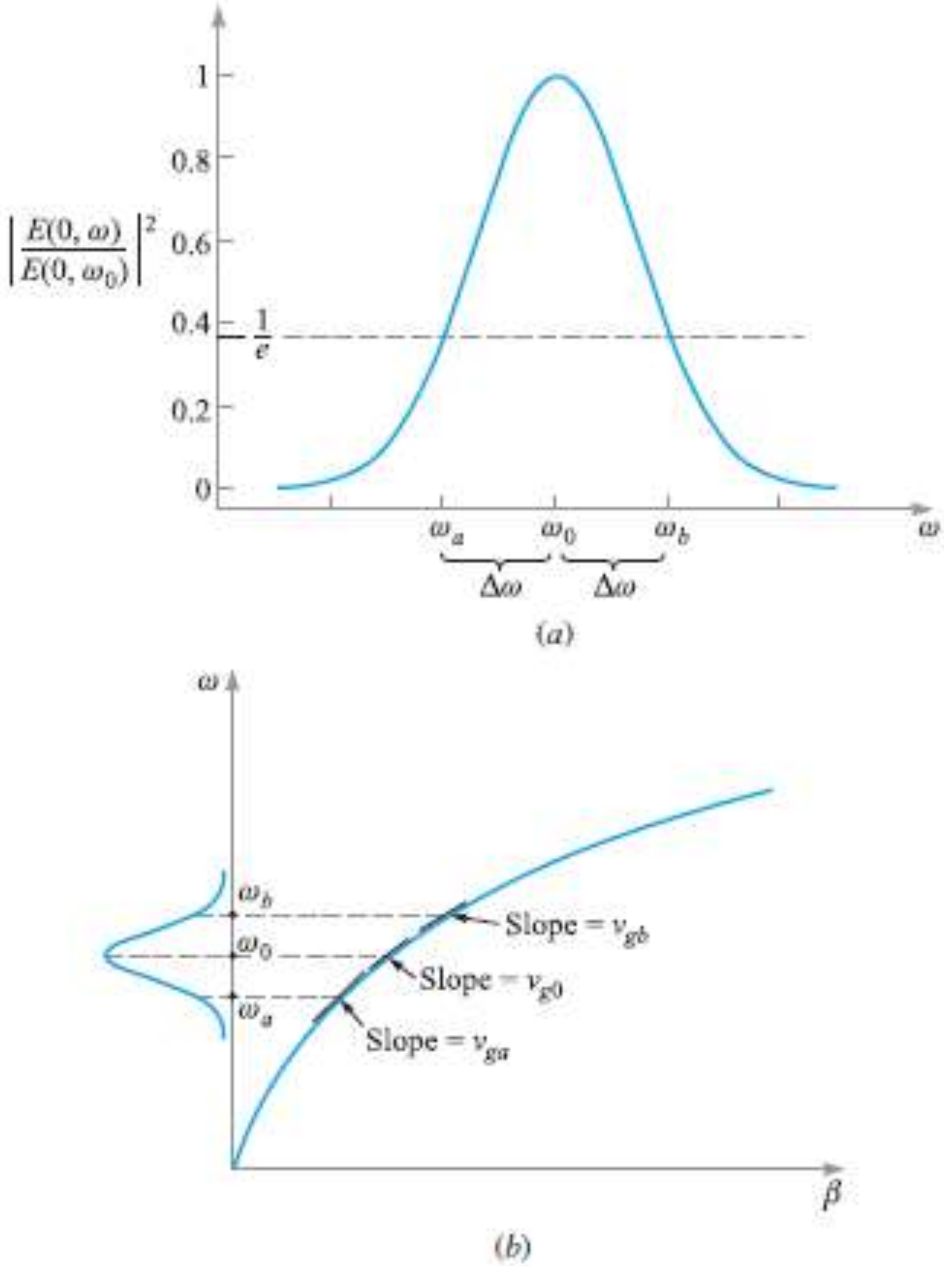



**Figure 12.14** (*a*) Normalized power spectrum of a Gaussian pulse, as determined from Eq. (86). The spectrum is centered at carrier frequency $\omega_0$ and has $1/e$ half-width, $\Delta\omega$. Frequencies $\omega_a$ and $\omega_b$ correspond to the $1/e$ positions on the spectrum. (*b*) The spectrum of Figure 12.14*a* as shown on the $\omega$-$\beta$ diagram for the medium. The three frequencies specified in Figure 12.14*a* are associated with three different slopes on the curve, resulting in different group delays for the spectral components.

Each frequency component propagates with *different velocity (GROUP DELAYS).*

The difference in arrival times with reference to $v_{g_o}$ is:

$$\Delta\tau = \frac{z}{v_{g_b}} - \frac{z}{v_{g_o}} = z\left(\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_b} - \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_o}\right) \rightarrow (1)$$

$\Delta\tau$ : pulse delay (pulse broadening)

Expanding $\beta(\omega)$ in its Taylor series around $\omega_o$

$$\beta(\omega) = \beta(\omega_o) + (\omega-\omega_o)\underbrace{\left.\frac{d\beta}{d\omega}\right|_{\omega_o}}_{\beta_1} + \frac{1}{2}(\omega-\omega_o)^2\underbrace{\left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_o}}_{\beta_2} + \left(O^3\right)$$

$\beta_2$ : Dispersion parameter in $\left(\frac{s^2}{m}\right)$.

So:

$$\beta(\omega) = \beta(\omega_o) + (\omega-\omega_o)\beta_1 + \frac{1}{2}(\omega-\omega_o)^2\beta_2 \rightarrow (2)$$

differentiating

$$\frac{d\beta}{d\omega} = \beta_1 + (\omega-\omega_o)\beta_2 \rightarrow (3)$$

Subs. (3) into (1) and simplifying to obtain the ***pulse delay (pulse broadening)***

$$\Delta\tau = \Delta\omega\beta_2 z = \frac{\beta_2 z}{T}$$

where

$$\Delta\omega = \omega_b - \omega_o = \frac{1}{T}$$

If the initial pulse width is very short compared to $\Delta\tau$, then the broadened pulse width at location $z$ will be simply $\Delta\tau$. If the initial pulse width is comparable to $\Delta\tau$, then the pulse width at $z$ can be found with the help of:

$$T_{new} = \sqrt{(T_{old})^2 + (\Delta\tau)^2}$$

$T_{old}$ : Original pulse width before propagating in dispersive media.

$T_{new}$ : Pulse width after propagating to location z.

Dispersion parameter units are in general time$^2$/distance, that is, pulse spread in time per unit spectral bandwidth, per unit distance. In optical fibers, for example, the units most commonly used are picoseconds$^2$/kilometer (psec$^2$/km).

**EXAMPLE 12.11**

An optical fiber link is known to have dispersion $\beta_2 = 20\ \text{ps}^2/\text{km}$. A Gaussian light pulse at the input of the fiber is of initial width $T = 10$ ps. Determine the width of the pulse at the fiber output if the fiber is 15 km long.

***Solution.*** The pulse spread will be

$$\Delta\tau = \frac{\beta_2 z}{T} = \frac{(20)(15)}{10} = 30\ \text{ps}$$

So the output pulse width is

$$T' = \sqrt{(10)^2 + (30)^2} = 32\ \text{ps}$$