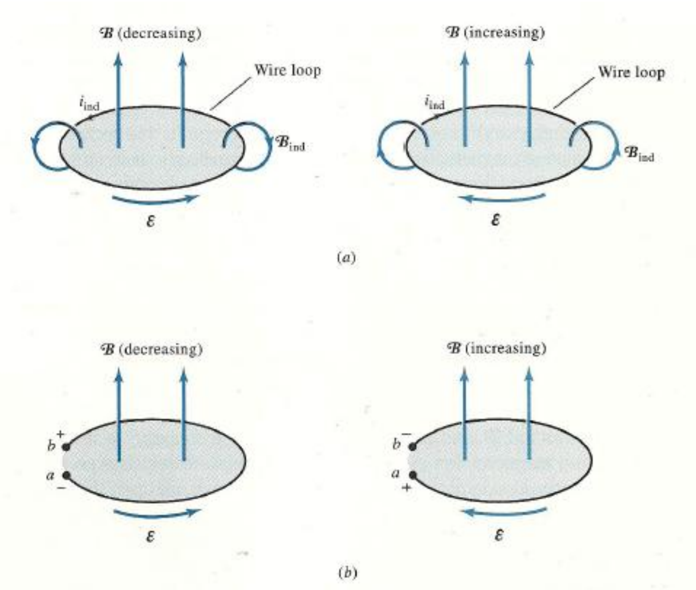
Time varying fields and Maxwell's equations

Chapter 9

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

- A time varying magnetic field produces (induces) a current in a closed loop of wire.
- The time varying magnetic field $\mathbf{B}(t)$ is said to induce an electromotive force (emf) in the loop and this emf drives the current.
- Electromotive force emf induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.



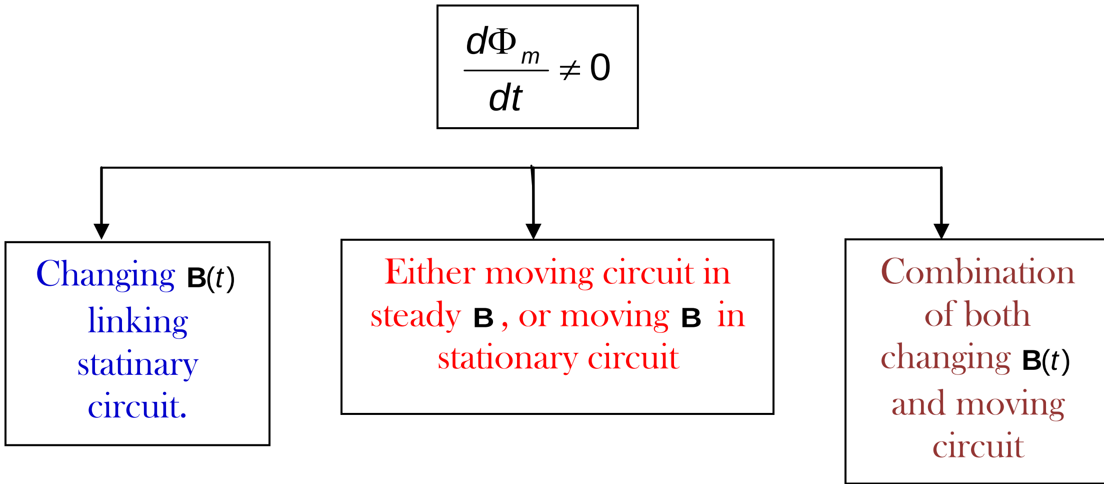
- ✓ **Lenz's law:** The direction of the emf is such that the magnetic flux generated by the induced current opposes the change in the original flux.

$$emf = -\frac{d\Phi_m}{dt}$$

$$emf = \oint_C \mathbf{E} \bullet \mathbf{dl}$$

$$\Phi_m = \iint_S \mathbf{B} \bullet \mathbf{ds}$$

- ✓ Fingers indicate the direction of C and the thumb indicate direction of $\mathbf{ds}$.
- ✓ Change in flux does not equal zero in the following cases:



If the closed path consists of N-turn filamentary conductor then

$$emf = -N\frac{d\Phi_m}{dt}$$

The general form of Farady's law is:

$$emf = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\iint_S \mathbf{B} \bullet \mathbf{ds} = \oint_C \mathbf{E} \bullet \mathbf{dl}$$

$$emf = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\iint_S \mathbf{B} \bullet \mathbf{ds} = \oint_C \mathbf{E} \bullet \mathbf{dl}$$

Rewrite:

$$\oint_C \mathbf{E} \bullet \mathbf{dl} = -\frac{d}{dt}\iint_S \mathbf{B} \bullet \mathbf{ds} = -\iint_S \frac{\partial}{\partial t}(\mathbf{B} \bullet \mathbf{ds}) \quad \oint_C \mathbf{E} \bullet \mathbf{dl} = -\iint_S \left(\frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds} + \mathbf{B} \bullet \frac{\partial \mathbf{ds}}{\partial t}\right)$$

If the circuit is stationary then $\frac{\partial \mathbf{ds}}{\partial t} = 0$, so

$$\oint_C \mathbf{E} \bullet \mathbf{dl} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds}$$

Obtaining the point form using Stokes Theorem;

$$\oint_C (\nabla \times \mathbf{E}) \bullet \mathbf{ds} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{ds}$$

$$\underbrace{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

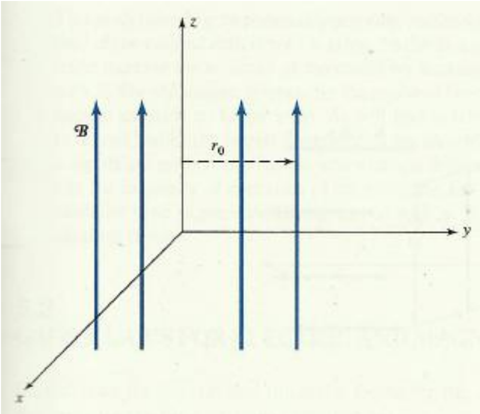
Point form of Faradays Law, or Maxwells first equation for time varying fields

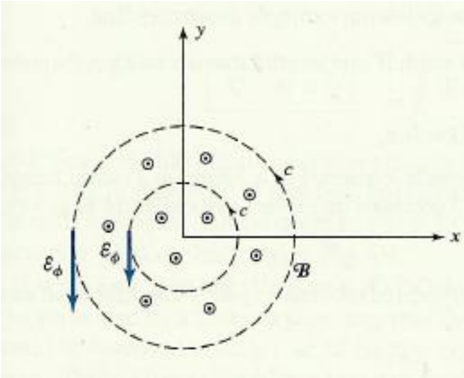
Example 6: Suppose a time varying magnetic field is defined in space (in a cylindrical coordinate system as:

$$\mathbf{B} = \begin{cases} B_o \sin(\omega t)\hat{\mathbf{a}}_z, & \rho \le \rho_o \\ 0, & \rho > 0 \end{cases}$$

a) Determine the induced electric field via Faradyay's law.
b) Show that the electric field intensity satisfies the point form of Faraday's law.

Solution:





a)

$$\mathbf{E} = E_\varphi(\rho)\hat{\mathbf{a}}_\varphi$$

$$\Phi_m = \iint_S \mathbf{B} \bullet \mathbf{ds} = \int_{\varphi=0}^{2\pi}\int_{\rho=0}^{\rho} B_o \sin(\omega t)\rho d\varphi d\rho$$

$$= \left\{\pi\rho^2 \omega B_o \sin(\omega t), \quad \rho \le \rho_o\right.$$

$$emf = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\iint_S \mathbf{B} \bullet \mathbf{ds}$$

$$= \left\{-\pi\rho^2 \omega B_o \cos(\omega t), \quad \rho \le \rho_o\right.$$

$$= \oint_C \mathbf{E} \bullet \mathbf{dl}$$

But:

$$\oint_C \mathbf{E} \bullet \mathbf{dl} = \int_{\varphi=0}^{2\pi} E_\varphi \rho d\varphi = 2\pi\rho E_\varphi$$

So:

$$E_\varphi = \left\{-\frac{\rho\omega B_o}{2}\cos(\omega t), \quad \rho \le \rho_o\right.$$

b)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$-\omega B_o \cos(\omega t)\hat{\mathbf{a}}_z = -\omega B_o \cos(\omega t)\hat{\mathbf{a}}_z$ and Maxwell's equation is satisfied.

MOTIONAL EMF

Moving Contours (Motional emf)

Consider the following device:

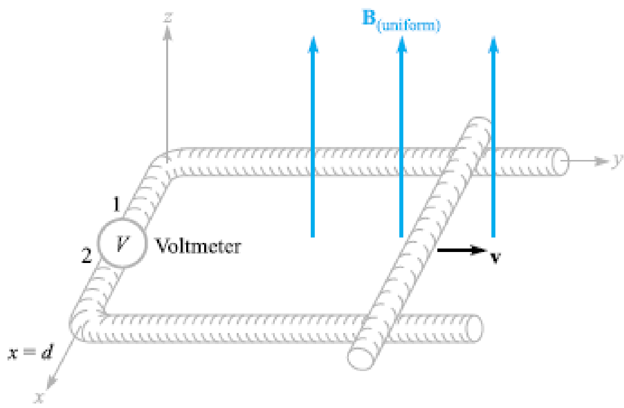


Figure 9.1 An example illustrating the application of Faraday's law to the case of a constant magnetic flux density B and a moving path. The shorting bar moves to the right with a velocity **v**, and the circuit is completed through the two rails and an extremely small high-resistance voltmeter. The voltmeter reading is $V_{12} = -Bvd$.

Method One: The magnetic flux linking S is:

$$\Phi_m = \iint_S \mathbf{B} \bullet \mathbf{ds} = Byd$$

$$emf = -\frac{d\Phi_m}{dt} = -Bd\frac{dy}{dt} = -Bdv$$

Method Two: The force on charge q moving at a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is:

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$$

The motional electric field intensity is:

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{v} \times \mathbf{B}$$

Then the motional emf produced by the moving conductor is:

$$emf = \oint_C \mathbf{E}_m \bullet \mathbf{dl} = \oint_C (\mathbf{v} \times \mathbf{B}) \bullet \mathbf{dl}$$

$$\oint_C (\mathbf{v} \times \mathbf{B}) \bullet \mathbf{dl} = \int_d^0 (v\hat{\mathbf{a}}_y \times B\hat{\mathbf{a}}_z) \bullet dx\hat{\mathbf{a}}_x$$

$$= -Bdv$$

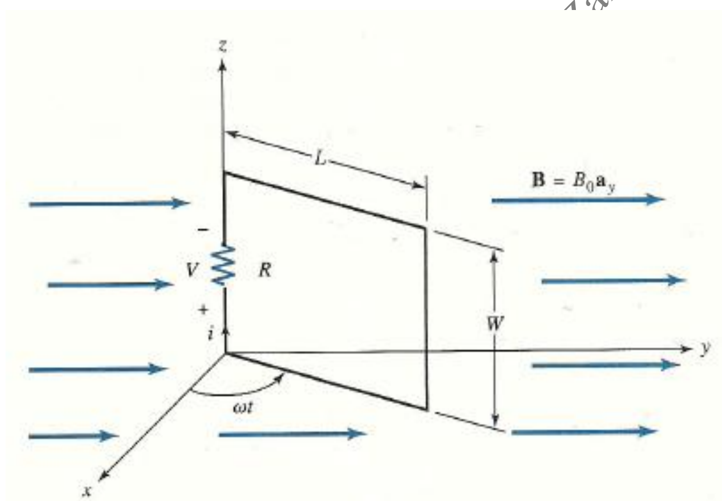
C and S satisfies the RHR.

If the circuit is moving in a changing magnetic field then the general form of Faraday's law is:

$$emf = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\iint_S \mathbf{B}\bullet\mathbf{ds}$$

$$= -\iint_S \frac{\partial\mathbf{B}}{\partial t}\bullet\mathbf{ds} + \oint_C (\mathbf{v}\times\mathbf{B})\bullet\mathbf{dl}$$

Example 7: consider a wire loop that is rotating in the presence of a dc magnetic field given by $\mathbf{B} = B_o\hat{\mathbf{a}}_y$ as shown. The loop have a resistor R inserted in it, and rotates at a radian frequency of ω and lies in the xz plane at $t = 0$. Determine the current induced in the loop.



$$\boldsymbol{B} = B_o\hat{\boldsymbol{a}}_y$$

$$\phi = \int_S \boldsymbol{B}\cdot\boldsymbol{ds} = \int_{\rho=0}^{L}\int_{z=0}^{W} B_o\hat{\boldsymbol{a}}_y\cdot dz d\rho\hat{\boldsymbol{a}}_\varphi$$

$$= LWB_o\cos(\omega t)$$

$$emf = -\frac{d}{dt}(LWB_o\cos(\omega t))$$

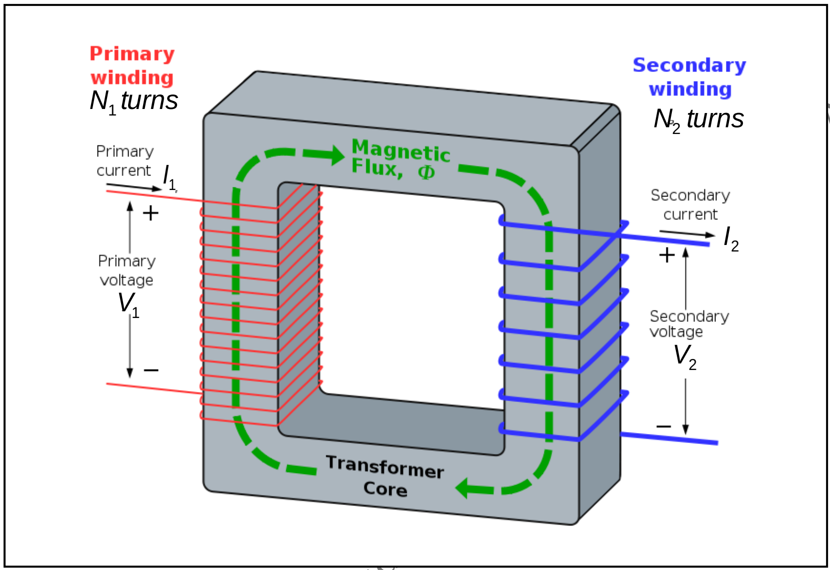
$$= \omega LWB_o\sin(\omega t)$$

$$i = \frac{emf}{R} = \frac{\omega LWB_o}{R}\sin(\omega t)$$

APPLICATIONS OF FARADAY'S LAW

- Transformers.
- Recent applications, includes Maglev and Witricity.
- Older applications include inductive heating.

Transformers



For an ideal Transformer

$$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$

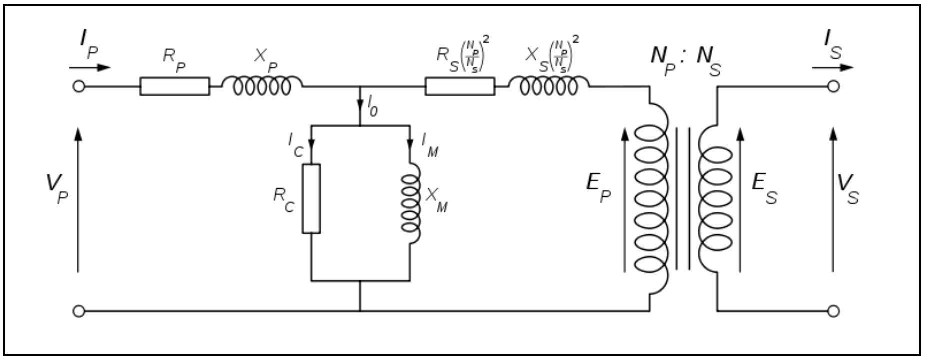
$$\frac{v_2}{v_1} = \frac{N_2}{N_1}$$

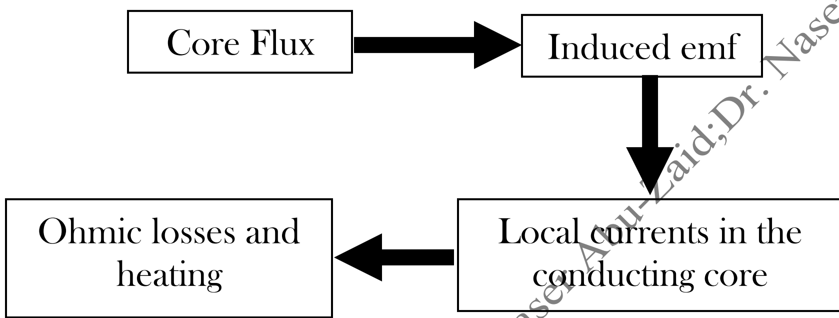
$$R_{eff} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Non idealities occur due to:

- ✓ Leakage flux
- ✓ Finite inductances
- ✓ Non-zero winding resistances
- ✓ Hysteresis
- ✓ Eddy currents loss

Usually approximate circuit for transformer is used.





DISPLACEMENT CURRENT AND MAXWELL'S EQUATIONS

For static fields, Maxwell's equations are given by:

$$\nabla \times \mathbf{E} = 0 \rightarrow (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} \rightarrow (2)$$

$$\nabla \bullet \mathbf{D} = \rho_v \rightarrow (3)$$

$$\nabla \bullet \mathbf{B} = 0 \rightarrow (4)$$

The first equation is already modified for time varying fields.

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \rightarrow (1')$$

The third and fourth equations remain the same.

What about the second equation?

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Taking the divergence of both sides

$$\nabla \bullet (\nabla \times \mathbf{H}) = 0 = \nabla \bullet \mathbf{J}$$

A contradiction

$$\nabla \bullet \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

Add an arbitrary term

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$$

Repeating the previous process

$$\nabla \bullet (\nabla \times \mathbf{H}) = 0 = \nabla \bullet \mathbf{J} + \nabla \bullet \mathbf{G}$$

So

$$-\nabla \bullet \mathbf{J} = \nabla \bullet \mathbf{G} = \frac{\partial \rho_v}{\partial t} = \frac{\partial \nabla \bullet \mathbf{D}}{\partial t}$$

$$= \nabla \bullet \frac{\partial \mathbf{D}}{\partial t}$$

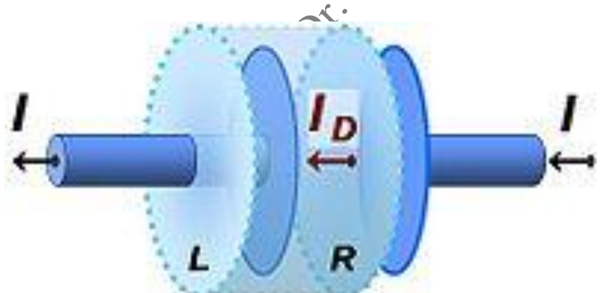
And Amper's law is modified to be

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Integrating over an open surface S enclosed by C and using Stoke's theorem:

$$\oint_C \mathbf{H} \bullet \mathbf{dl} = \iint_S \mathbf{J} \bullet \mathbf{ds} + \iint_S \frac{\partial \mathbf{D}}{\partial t} \bullet \mathbf{ds}$$

Illustration:



An electrically charging capacitor with an imaginary cylindrical surface surrounding the left-hand plate. Right-hand surface R lies in the space between the plates and left-hand surface L lies to the left of the left plate. No conduction current enters cylinder surface R, while current I leaves through surface L. Consistency of Ampère's law requires a displacement current $I_D = I$ to flow across surface R. ©Taken from wikipedia

Example 8: compare the conduction and displacement current densities in copper $\left(\varepsilon=\varepsilon_o, \mu=\mu_o, \sigma=5.8\times10^{7}\right)$ at a frequency of $1MHz$. Repeat for Teflon, which has $\left(\varepsilon=2.1\varepsilon_o, \mu=\mu_o, \sigma=3\times10^{-8}\right)$ at $1MHz$. Assume $E = E_o\sin(\omega t)$

Solution:

$$J_c = \sigma E = \sigma E_o \sin(\omega t)$$

$$J_d = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t} = \omega\varepsilon E_o \sin(\omega t)$$

$$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon}$$

For copper :

$$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{5.8\times10^{7}}{2\pi\times10^{6}\times8.854\times10^{-12}} = 10^{12}$$

For Teflon

$$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{3\times10^{-8}}{2\pi\times10^{6}\times2.1\times8.854\times10^{-12}} = 2.57\times10^{-4}$$

MAXWELL'S EQUATIONS AND BOUNDARY CONDITIONS

For time varying fields, Maxwell's equations are given by:

Point Form	Integral Form	Significance
$\nabla\times\mathbf{E} = -\frac{\partial B}{\partial t}$	$\oint_C \mathbf{E}\bullet\mathbf{dl} = -\frac{d}{dt}\iint_S \mathbf{B}\bullet\mathbf{ds}$	Faraday's Law
$\nabla\times\mathbf{H} = \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$	$\oint_C \mathbf{H}\bullet\mathbf{dl} = \iint_S \mathbf{J}\bullet\mathbf{ds} + \iint_S \frac{\partial\mathbf{D}}{\partial t}\bullet\mathbf{ds}$	Amper's Law
$\nabla\bullet\mathbf{D} = \rho_v$	$\oint\!\!\oint_S \mathbf{D}\bullet\mathbf{ds} = Q$	Gauss's Law for electrostatics
$\nabla\bullet\mathbf{B} = 0$	$\oint\!\!\oint_S \mathbf{B}\bullet\mathbf{ds} = 0$	No isolated magnetic charge (Gauss's Law for magnetostatics)

- Sources: Charge and Current Density.
- Auxiliary equations (constitutive relations):

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

- The conduction current density:

$$\mathbf{J} = \sigma \mathbf{E}$$

The boundary conditions for physical media, $(\mathbf{K} = 0)$

$$E_{t1} = E_{t2}$$

$$H_{t1} = H_{t2}$$

$$D_{n1} - D_{n2} = \rho_s$$

$$B_{n1} = B_{n2}$$

CONDITIONS FOR A PERFECT CONDUCTOR

Ohm's Law:

$$\sigma \to \infty \quad \Rightarrow \quad \mathbf{E} = 0 \quad \textit{inside a perfect conductor}$$

So;

$$D = 0 \quad \textit{inside a perfect conductor}$$

Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \quad \Rightarrow \quad \frac{\partial B}{\partial t} = 0$$

So;

$$B(t) = 0 \quad \textit{inside a perfect conductor} \ (\textit{no time varing } B \textit{ may exist})$$

$$\mathbf{H} = 0 \ \textit{inside a perfect conductor}$$

Amper's law:

$$\mathbf{J} = 0 \ \textit{inside a perfect conductor}$$

So, if region II is a perfect conductor then

$$E_{t1} = 0$$

$$H_{t1} = K \quad \text{or} \quad (\mathbf{H}_{t1} = \mathbf{K} \times \hat{\mathbf{a}}_n) \ \textit{outward normal}$$

$$D_{n1} = \rho_s$$

$$B_{n1} = 0$$

Example 9: Show that the following vector fields in free space satisfy all of Maxwell's equations.

$$\mathbf{E} = E_{xo}\cos(\omega t - k_o z)\hat{\mathbf{a}}_x$$

$$\mathbf{H} = \frac{E_{xo}}{\eta_o}\cos(\omega t - k_o z)\hat{\mathbf{a}}_y$$

Where η_o, E_{xo}, k_o are constants.

Solution:

a) Faraday's law

$$\underbrace{\nabla \times \boldsymbol{E}}_{1} = \frac{\partial E_x}{\partial z}\hat{\boldsymbol{a}}_y = \underbrace{k_o E_{xo} sin(\omega t - k_o z)\hat{\boldsymbol{a}}_y}_{1} = \underbrace{-\frac{\partial B}{\partial t}}_{2}$$

$$= \underbrace{\frac{\mu_o E_{xo}}{\eta_o}\omega_o sin(\omega t - k_o z)\hat{\boldsymbol{a}}_y}_{2}$$

1=2 for Faraday's law to be satisfied, this implies;

$$k_o = \frac{\omega \mu_o}{\eta_o}$$

b) Amper's law

$$\underbrace{\nabla \times \boldsymbol{H}}_{1} = -\frac{\partial H_y}{\partial z}\hat{\boldsymbol{a}}_x = \underbrace{\frac{-k_o E_{xo}}{\eta_o} sin(\omega t - k_o z)\hat{\boldsymbol{a}}_x}_{1} = \underbrace{\frac{\partial \boldsymbol{D}}{\partial t}}_{2}$$

$$= -\underbrace{\omega_o \varepsilon_o E_{xo} sin(\omega t - k_o z)\hat{\boldsymbol{a}}_x}_{2}$$

1=2 for Amper's law to be satisfied, this implies;

$$k_o = \omega \varepsilon_o \eta_o$$

c) Gauss law

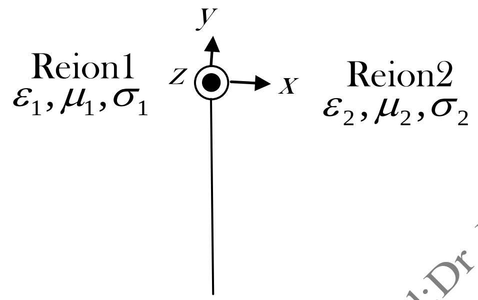
$$\nabla \cdot \boldsymbol{D} = \rho_v = 0 = \varepsilon_o\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] = 0 \quad \text{(Satisfied)}$$

d) The other Gauss law

$$\nabla \cdot \boldsymbol{B} = 0 = \mu_o\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right] = 0 \quad \text{(Satisfied)}$$

Example 10: At the interface between two regions as shown in the figure, find the magnetic field intensity vector at $x = 0$ if:

$$\mathbf{H}_1 = \alpha\,\hat{\mathbf{a}}_x + \beta\,\hat{\mathbf{a}}_y + \delta\,\hat{\mathbf{a}}_z \quad @\, x = 0$$



Solution:

$$\boldsymbol{H}_{t1} = \beta\hat{\boldsymbol{a}}_y + \delta\hat{\boldsymbol{a}}_z$$
$$\boldsymbol{H}_{n1} = \alpha\hat{\boldsymbol{a}}_x$$
$$\boldsymbol{H}_{t1} = \boldsymbol{H}_{t2} = \beta\hat{\boldsymbol{a}}_y + \delta\hat{\boldsymbol{a}}_z$$
$$\boldsymbol{B}_{n1} = \mu_1\boldsymbol{H}_{n1} = \mu_1\alpha\hat{\boldsymbol{a}}_x$$
$$\boldsymbol{B}_{n2} = \boldsymbol{B}_{n1} = \mu_1\alpha\hat{\boldsymbol{a}}_x$$
$$\boldsymbol{H}_{n2} = \frac{\boldsymbol{B}_{n2}}{\mu_2} = \frac{\mu_1}{\mu_2}\alpha\hat{\boldsymbol{a}}_x$$

So;

$$\boldsymbol{H}_2 = \boldsymbol{H}_{t2} + \boldsymbol{H}_{n2} = \beta\hat{\boldsymbol{a}}_y + \delta\hat{\boldsymbol{a}}_z + \frac{\mu_1}{\mu_2}\alpha\hat{\boldsymbol{a}}_x$$

THE RETARDED POTENTIALS

- Time-varying potentials, usually called *retarded potentials*.
- Remember that the scalar electric potential V may be expressed in terms of a static charge distribution;

$$V = \int_{\text{vol}} \frac{\rho_v\,dv}{4\pi\epsilon R} \quad \text{(static)}$$

- The vector magnetic potential may be found from a current distribution which is constant with time;

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}\, dv}{4\pi R} \quad \text{(dc)}$$

The differential equations satisfied by V;

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{(static)}$$

and for $\boldsymbol{A}$,

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \quad \text{(dc)}$$

Having found V and $\boldsymbol{A}$, the fundamental fields are then simply obtained by using the gradient,

$$\mathbf{E} = -\nabla V \quad \text{(static)}$$

Or the curl,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(dc)}$$

Time-varying potentials (defined in a way) which are consistent with time varying Maxwell's equations; leads to:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Later, we will find that any electromagnetic disturbance will travel at a *finite velocity of*

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

through any homogeneous medium described by ϵ and μ. In the case of free space, this velocity turns out to be the velocity of light, approximately

$$c \approx 3 \times 10^8 (m/s)$$

It is logical, then, to suspect that the **potential at any point is due not** to the value of the charge density at some distant point **at the same instant**, but to its value at ***some previous time***, because ***the effect propagates at a finite velocity***.

Thus;

$$V = \int_{vol} \frac{[\rho_v]}{4\pi\epsilon R} dv$$

Where $[\rho_v]$ indicates that every t appearing in the expression for ρ_v has been replaced by a *retarded* time,

$$t' = t - \frac{R}{v}$$

Thus, if the charge density throughout space were given by

$$\rho_v = e^{-r} \cos \omega t$$

Then

$$[\rho_v] = e^{-r} \cos\left[\omega\left(t - \frac{R}{v}\right)\right]$$

Where R is the distance between the differential element of charge being considered and the point at which the potential is to be determined.

The retarded vector magnetic potential is given by

$$\mathbf{A} = \int_{vol} \frac{\mu[\mathbf{J}]}{4\pi R} dv$$

Summary:

- Use the distribution of ρ_v and $\boldsymbol{J}$ to determine V and $\boldsymbol{A}$ by applying:

$$V = \int_{vol} \frac{[\rho_v]}{4\pi\epsilon R} dv$$

$$\mathbf{A} = \int_{vol} \frac{\mu[\mathbf{J}]}{4\pi R} dv$$

- Electric and magnetic fields are then obtained by applying:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$